QUANTUM YANG-MILLS THEORY IN TWO DIMENSIONS: EXACT VERSUS PERTURBATIVE

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ABSTRACT. The standard Feynman diagrammatic approach to quantum field theories assumes that perturbation theory approximates the full quantum theory at small coupling even when a mathematically rigorous construction of the latter is absent. On the other hand, two-dimensional Yang-Mills theory is a rare (if not the only) example of a nonabelian (pure) gauge theory whose full quantum theory has a rigorous construction. Indeed, the theory can be formulated via a lattice approximation, from which Wilson loop expecation values in the continuum limit can be described in terms of heat kernels on the gauge group. It is therefore fundamental to investigate how the exact answer for 2D Yang-Mills compares with that of the continuum perturbative approach, which a priori are unrelated. In this paper, we provide a mathematically rigorous formulation of the perturbative quantization of 2D Yang-Mills, and we consider Wilson loop expectation values on $\mathbb{R}^2$ and S^2 in both holomorphic gauge and in Coulomb gauge with respect to a general metric. We establish the equivalence of these two gauges and that both are independent of the choice of gauge-fixing metric. Additionally, using holomorphic gauge, we discuss partial results showing agreement between the asymptotics of perturbative and lattice Wilson loop expectations. Our work therefore presents fundamental progress in confirming the paradigm that continuum perturbation theory accurately captures the asymptotics of the continuum limit of the lattice theory.

CONTENTS

1. Introduction

Yang-Mills theory provides a theoretical framework for describing the physics of elementary particles and has profoundly impacted the study of partial differential equations and differential topology. The classical (Euclidean) Yang-Mills action can be written as

$$S_{YM}(A) = \frac{1}{2e^2} \int_\Sigma \langle F_A \wedge *F_A \rangle, \tag{1.1}$$

where F_A is the gauge-field strength, $\langle \cdot, \cdot \rangle$ is an ad-invariant inner product on the Lie algebra of the gauge group, e is a coupling constant, and Σ is the underlying space assumed to be a smooth, orientable Riemannian manifold. In the quantum theory, this classical action is inserted into a (formal) path integral, from which one can compute various physical quantities in terms of a Feynman diagrammatic expansion. The process of evaluating and understanding such Feynman diagrams is what leads to many of the basic features of quantum Yang-Mills theory, such as perturbative renormalizability [41, 13, 15] and asymptotic freedom [25, 35].

On the other hand, quantum Yang-Mills theory can also be formulated on a lattice, whereby one obtains a mathematically rigorous, nonperturbative approach that avoids the formal aspects of the continuum theory outlined above. Indeed, working on a lattice ensures that all quantities can be computed in terms of well-defined finite-dimensional integrals. Here, one has to introduce a suitable discretization of the action (1.1) the details of which we will return to later. But as one is ultimately interested in a theory that extends all the way down to the relevant microscopic scales, one would like to take a continuum limit in which the lattice spacing becomes finer and finer.

Supposing this to be achieved, we obtain two independent, a priori distinct constructions of quantum Yang-Mills theory. While quite different, both the perturbative methods of Feynman diagrams and the numerical simulations of lattice methods have yielded spectacular agreement with experimental data in various settings [28, 34, 36]. Naturally then, one should consider how these two methods compare at the level of precise mathematics. Specifically, since one regards the continuum formulation as perturbative and the lattice formulation as nonperturbative, one expects in the limit of small coupling that the two formulations should somehow converge. For emphasis, we state this as the following

Question: *For Yang-Mills theory, what is the relationship between the perturbative results obtained in the continuum formulation and the nonperturbative results obtained from the continuum limit of the lattice formulation, as the coupling constant is sent to zero?*

This paper is an investigation into this basic question, for which we are unaware of any prior rigorous work by the mathematical community.

A priori, our question is well-posed only if we know how to take the continuum limit of the lattice formulation. This is a very difficult problem in dimensions three and four, for which there exists old work by Balaban [6, 7] that is unfortunately not easily accessible. However, we are in the fortunate situation that in two dimensions, quantum Yang-Mills theory has a well-known and elegant lattice continuum limit due to Migdal [29] and Witten [43]. The aim of this paper is to compare this continuum limit with perturbative two-dimensional Yang-Mills theory.

Our approach is as follows. The quantities of interest to us are expectation values of Wilson loop observables, which are the basic gauge-invariant observables of any gauge theory. Given an oriented closed curve γ and a conjugation-invariant function f on the gauge group G of our theory, we obtain the Wilson loop observable $W_{f,\gamma}$ which takes a connection A, computes the holonomy of A about γ, and applies f to this group-valued element:

$$W_{f,\gamma}(A) = f(\mathrm{hol}_\gamma(A)).$$

We can compute the expectation value of $W_{f,\gamma}(A)$ exactly using the continuum of the lattice approach or perturbatively using the methods of Feynman diagrams:

$$\langle W_{f,\gamma}\rangle := \text{(exact) expectation value}$$
$$\langle W_{f,\gamma}\rangle_{pert} := \text{perturbative expectation value}.$$

The exact expectation value is defined mathematically precisely in Section 2; for the perturbative expectation, defined in Section 3, a few clarifying remarks are in order. We will primarily be considering the special case $\Sigma = S^2$. This case is natural for several reasons. For the perturbation theory, a compact underlying space is natural since this eliminates infrared divergences. Moreover, S^2 is especially simplifying because then there is a unique minimal Yang-Mills connection modulo gauge-equivalence, namely the trivial connection[1]. For Σ of higher genus, the presence of a continuous moduli of flat connections makes the perturbation theory more involved. Thus, we take $\Sigma = S^2$ and $\langle W_{f,\gamma}\rangle_{pert}$ denotes the Feynman diagrammatic expansion about the trivial connection. We also consider $\Sigma = \mathbb{R}^2$ which (in the appropriate instances) we can regard as the limit of S^2 when the area of the latter is sent to infinity. We refer to this limiting procedure as "decompactification".

Next, note that in two dimensions, the Hodge star operator $*$ appearing in the integral (1.1) is specified entirely in terms of an area form $d\sigma$ on Σ (and not on a full metric tensor). It follows that the coupling constant $\lambda_0 = e^2$ has dimensions of inverse area. (In other words, scaling the area form by ℓ^2 and the coupling constant λ_0 by ℓ^{-2} preserves the action.) Thus, we define the dimensionless coupling constant

$$\lambda = \lambda_0|\Sigma| \tag{1.2}$$

where $|\cdot|$ denotes area with respect to $d\sigma$. Thus, $\langle W_{f,\gamma}\rangle$ is a function of λ while $\langle W_{f,\gamma}\rangle_{pert}$ is a formal power series in λ.

Finally and quite crucially, the perturbative expectation value $\langle W_{f,\gamma}\rangle_{pert}$ requires a suitable gauge-fixing procedure to be introduced. The most natural choice of gauge to consider is Coulomb gauge (also known as Landau gauge). Here, one chooses an auxiliary metric $g = g_{ij}$ and imposes the gauge-fixing condition $d^*A = 0$ to eliminate longitudinal modes. The geometric nature of Coulomb gauge makes it applicable for arbitrary Σ. In the case of S^2 however, we have available another choice of "gauge", namely holomorphic gauge. This gauge is referred to as (Euclidean) light-cone gauge in the physics literature. Here, writing A as $A_z dz + A_{\bar{z}} d\bar{z}$ in terms of holomorphic and anti-holomorphic components (with respect to a chosen auxiliary conformal structure), holomorphic gauge imposes the condition $A_{\bar{z}} = 0$. While the interpretation of this condition as a gauge-fixing condition requires

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[1]We will be focusing only on topologically trivial bundles. This is not a real restriction, see Remark 2.5.

some additional analysis [32], Feynman diagrams can be meaningfully generated from this ansatz notwithstanding.

Thus, we consider in this paper

$$\langle W_{f,\gamma} \rangle_{pert} = \langle W_{f,\gamma} \rangle_C \ \text{or} \ \langle W_{f,\gamma} \rangle_{hol} \,,$$

the perturbative expectation corresponding to Coulomb gauge or holomorphic gauge, respectively, which are defined mathematically precisely in Section 3. Each involves the choice of an auxiliary metric compatible with the given area form on S^2 (for holomorphic gauge, the data of a conformal structure and the given area form is equivalent to a compatible metric).

Our results can be summarized as follows, which we represent pictorially alongside additional results from [32, 33] in Figure 1:

Theorem 1. *Consider Yang-Mills theory on $(S^2, d\sigma)$ with arbitrary compact gauge group G. Pick any compatible metric for use as a gauge-fixing metric.*

(i) *$\langle W_{f,\gamma} \rangle_C$, defined using an appropriate regularization scheme, is finite without any need for counterterms and is independent of the choice of gauge-fixing metric. Moreover, $\langle W_{f,\gamma} \rangle_C$ is invariant under area-preserving diffeomorphisms in the sense of Corollary 3.17.*

(ii) *We have*

$$\langle W_{f,\gamma} \rangle_C = \langle W_{f,\gamma} \rangle_{hol} \tag{1.3}$$

In particular, $\langle W_{f,\gamma} \rangle_{hol}$ is also independent of the choice of gauge-fixing metric.

(iii) *Let γ be a simple closed curve. Then the series expansion of $\lim_{\lambda \to 0} \langle W_{f,\gamma} \rangle$ is explicitly given by a Gaussian integral over $\mathfrak{g}$ given by the formula (4.7). Moreover, this series differs from $\langle W_{f,\gamma} \rangle$ by exponentially small "instanton" corrections, see Remark 4.4.*

Conjecture 1. *We have*

$$\lim_{\lambda \to 0} \langle W_{f,\gamma} \rangle \sim \langle W_{f,\gamma} \rangle_{hol} \,, \tag{1.4}$$

Here, $\sim$ means that the left-hand side of (1.4) is described asymptotically by the series on the right-hand side[2].

Theorem 2. *Let γ be a simple closed curve. Then Conjecture 1 holds true up to second order in perturbation theory, i.e., up to $O(\lambda^3)$, in the decompactification limit $S^2 \to \mathbb{R}^2$.*

Except for (iii), the above statements and Conjecture 1 readily extend to arbitrary products of Wilson loop observables.

For (i), a regulator is needed since we will obtain divergent Feynman diagrams in the expansion defining $\langle W_{f,\gamma} \rangle_C$. We use a heat kernel regulator. That no counterterms are needed even as the regulator is removed is shown in Theorem 3.3 via explicit computations. Statement (ii), which proves the equivalence of Coloumb and holomorphic gauge (to all orders in perturbation theory), is surprising from a purely mathematical point of view since the constructions are very different. From a practical point of view, since holomorphic

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[2]In other words, $\psi(\lambda) \sim \sum_{n \geq 0} c_n \lambda^n$ if $\psi(\lambda) - \sum_{n=0}^{N} c_n \lambda^n = o(\lambda^N)$ as $\lambda \to 0$ for every N. We write $\lim_{\lambda \to 0}$ to emphasize that we are considering small λ asymptotics.

gauge is much more computationally feasible than Coulomb gauge, (ii) represents a dramatic simplification. Previously, this equivalence was explicitly checked to second order in special cases [8, 9]. Statement (iii) provides an explicit computation of the asymptotics of Wilson loop expectation of a simple closed curve to all orders in the coupling. Moreover, it shows explicitly how the exact answer and the perturbative answer differ through asymptotically zero instanton corrections. There has been previous work on (iii) in the physics literature [9, 10, 23], though we were unaware of them when we obtained (iii) independently. The result is somewhat mysterious, see Section 5 for some discussion. Theorem 2 provides partial confirmation of Conjecture 1, see Theorem 4.1 for a more precise statement.

Altogether, our results, still to be further completed by resolving Conjecture 1, corroborate the central tenant of quantum field theory which asserts that the Feynman diagrammatic perturbative expansion yields an asymptotic series for the exact expectation of observables:

$$\lim_{\lambda \to 0} \langle O \rangle \sim \langle O \rangle_{pert}. \tag{1.5}$$

Despite enormous efforts to place quantum field theory on firm mathematical foundations, establishing the fundamental consistency condition (1.5) in the context of gauge-theories appears to have been overlooked by the mathematical community.
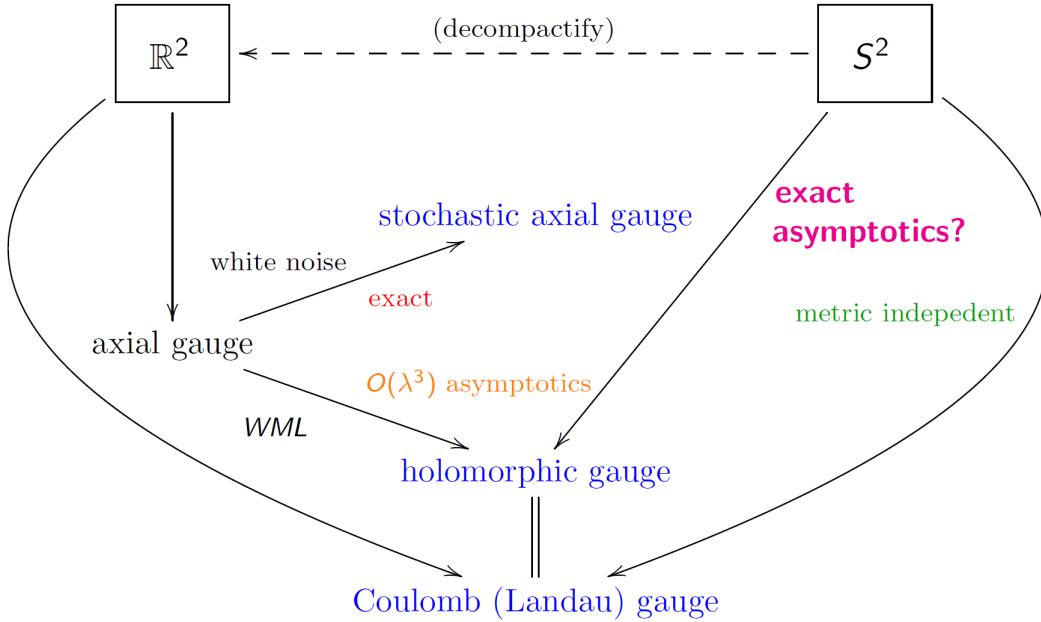

FIGURE 1. Roadmap of equivalences and results.

Figure 1 illustrates how our main results above and our subsequent work [32, 33] at present fit together. Results (i) and (ii) on S^2 are given by the rightmost arrow and the equality between holomorphic and Coulomb gauge in Figure 1. Moreover, when we decompactify

S^2 to $\mathbb{R}^2$, these results about Coulomb gauge and holomorphic gauge carry over to $\mathbb{R}^2$ if we define the latter gauges as the limit of those obtain from S^2 (i.e. take the propagators on $\mathbb{R}^2$ to be the limit of those obtained on S^2). Additionally, we can consider various axial gauges on $\mathbb{R}^2$ as we did in [32, 33]. Evaluating perturbative Wilson loop expectations in axial-gauge[3] provides additional settings in which to test (1.5). One source of ambiguity with axial gauge is that it needs a regulator. When one uses stochastic methods (i.e. white-noise analysis) to regulate the axial gauge, one obtains "stochastic axial-gauge", which actually decomposes a priori into two cases: partial axial-gauge and complete axial-gauge. In [33], we establish the equivalence between these two gauges, hence we have a single stochastic axial gauge listed in Figure 1. On the other hand, the Wu-Mandelstam-Liebrandt (WML) regulator leads to "generalized axial-gauge" (i.e. the connection has a vanishing component along a complexified direction), which is equivalent to holomorphic gauge on $\mathbb{R}^2$ [32].

The analysis of (1.5) for Wilson loops really splits into three cases:

(1) decompactify to $\mathbb{R}^2$ then compute asymptotics;
(2) compute asymptotics on S^2 then decompactify;
(3) compute asymptotics on S^2.

For case (1), we established (1.5) in the stochastic axial-gauge to all orders in perturbation theory [33]. Conjecture 1 therefore concerns cases (2) and (3) using the holomorphic gauge (this is denoted by the question in Figure 1). Remarkably, (1) and (2) are *inequivalent*, i.e. Wilson loop asymptotics computed using axial gauge on $\mathbb{R}^2$ and holomorphic gauge on $\mathbb{R}^2$ (the second is the decompactification of holomorphic gauge on S^2) differ, see Theorem 4.2. A physical explanation for this phenomenon is that the asymptotics on S^2 "remembers instantons" when decompactified, hence differing from the asymptotics computed after decompactification [9]. For case (2), Theorem 2 verifies Conjecture 1 to second order in perturbation theory for Wilson loops given by simple closed curves. This result is denoted by the $O(\lambda^3)$ arrow in Figure 1. Conjecture 1 in cases (2) and (3) can also be verified to all orders for highly symmetric configurations [32].

Our establishing the fundamental relation (1.5) in stochastic axial-gauge on $\mathbb{R}^2$ made use of the fact that Yang-Mills theory is (a) free in axial-gauge; (b) has a measure-theoretic interpretation in (complete) axial gauge. Thus, relating continuum Yang-Mills theory in stochastic axial-gauge to the lattice formulation is to be expected, though by no means trivial [17, 33]. (In fact, we know the analysis cannot be trivial because holomorphic gauge and stochastic axial-gauge, while both rendering Yang-Mills theory free, yield inequivalent Wilson loop expectation values). Thus, what makes Conjecture 1 interesting is that (a') there is no (bona fide) axial gauge on S^2, i.e. a real global direction on S^2 which we can gauge away; (b') holomorphic gauge does not have a measure-theoretic interpretation since the corresponding propagator does not define a positive-definite pairing. Consequently, relating holomorphic gauge to the asymptotics on the lattice is not at all obvious. Altogether, it remains to consider Conjecture 1 for cases (2) and (3) for general loops to all orders.

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[3]Axial gauge, like holomorphic gauge, while rendering two-dimensional Yang-Mills theory free, leads to a "perturbative" Feynman diagram expansion in the sense that the resulting expansion is a priori a formal one that does not require the existence of a measure, i.e. an honest expectation. Thus, investigating (1.5) is not vacuous in axial gauge [33].

Our paper is organized as follows. In Section 2, we discuss the lattice formulation of 2D Yang-Mills theory and describe how its continuum limit yields a rigorous construction of a Yang-Mills measure. The most important outcome of this is that one obtains concrete formulas for Wilson loop expectation values. In Section 3, we describe the entirely different methods of perturbative quantization of continuum Yang-Mills theory. Here we give a rapid, self-contained setup of formal perturbation theory in Coulomb gauge using the most direct procedure available: the Faddeev-Popov method. While many physics treatments of the Faddeev-Popov method use formal arguments to assert that it leads to a gauge-invariant construction, proving that this is so mathematically rigorously takes a fair amount of sophistication. For this, we use the Batalin-Vilkovisky method of quantization, which is powerful enough to handle the situation in which there are zero ghost modes (which we do find ourselves in since we quantize about a trivial connection). This is done in Section 3.3, which being rather technical, we isolate as an independent section on its own since its importance is of a conceptual, not computational nature. We also use the Batalin-Vilkovisky formalism to prove the equivalence of Coulomb gauge and holomorphic gauge in Section 3.4. Finally in Section 4, we use analytic and Lie theoretic tools to relate the asymptotics of exact Wilson loop expectation values to perturbative calculations. We finish with a discussion of our results and future directions. Since the methods of perturbative quantum field theory are not well-known to most mathematicians, our appendix provides background on Wick's Theorem so as to make this paper as self-contained as possible.

2. 2D Yang-Mills Measure

Euclidean quantum Yang-Mills theory, in any dimension, can be given a rigorous formulation on a lattice. As there is no canonical choice for how to discretize a continuum theory, many formulations are possible. In two dimensions it is most convenient to work with the one due to Witten [43], which extends the original approach of Migdal [29]. This Migdal-Witten formulation is invariant with respect to lattice subdivision, so that the continuum limit of taking the lattice spacing to zero is in some sense already inherent.

We recall this formulation (following [26]) in the general setting of when the underlying space is a closed, connected surface Σ endowed with an area form $\lambda_0 d\sigma$, where $d\sigma$ is some reference area form and λ_0 a coupling constant. Let G be a compact Lie group and Γ be a triangulation of Σ, that is, a finite set of edges (mappings of intervals $[0, 1]$ into Σ, injective on the interiors) joined at their vertices such that their complement consists of a disjoint union of faces homeomorphic to disks. Assign an arbitrary orientation to each edge. To these oriented edges e of Γ, we assign group valued elements $g_e \in G$, which form the basic variables of the discretized Yang-Mills theory. To each face F, we can compute its

area $|F|$ (with respect to $d\sigma$) and assign an orientation to the boundary ∂F of F, thereby inducing an orientation on all the edges which comprise it. Write $\partial F = \pm e_n \cdots \pm e_1$ as a concatenation of edges occurring in their natural cyclic order (well-defined up to cyclic permutation), where one has $\pm$ according to whether the preassigned orientation of the edge e_i is the same or opposite of that induced from ∂F. Define

$$g_{\partial F} = g_{e_n}^{\pm 1} \cdots g_{e_1}^{\pm 1},$$

the product of the group valued elements associated to ∂F with the appropriate powers. It is well-defined up to cyclic permutation of the factors and inversion.

Pick any bi-invariant metric on G. This is obtained from an ad-invariant metric on its Lie algebra $\mathfrak{g}$, which we denote by $\langle \cdot, \cdot \rangle$. One obtains an associated Laplace-Beltrami operator Δ on G. The heat kernel for Δ is given by the function $K_t(g)$, $t > 0$, which satisfies

$$(e^{-t\Delta/2}f)(h) = \int_G K_t(hg^{-1})f(g)dg \tag{2.1}$$

for all smooth functions f on G. Here dg denotes normalized Haar measure on G. The function $K_t(g)$ satisfies

$$K_t(g^{-1}) = K_t(g), \qquad K_t(gh) = K_t(hg), \qquad \text{for all } g, h \in G. \tag{2.2}$$

Assign the weight $K_{\lambda_0|F|}(g_{\partial F})$ to F. Properties (2.2) show that this weight is independent of the orientation of ∂F and the cyclic ordering of the edges in ∂F. The Yang-Mills measure associated to Γ is the measure

$$d\mu_{\Gamma,\lambda_0 d\sigma} = \Big(\prod_{F \in F(\Gamma)} K_{\lambda_0|F|}(g_{\partial F}) \Big) \prod_{e \in E(\Gamma)} dg_e, \tag{2.3}$$

on $G^{|E(\Gamma)|}$, where $E(\Gamma)$ and $F(\Gamma)$ are the set of edges and faces of Γ, respectively. The Yang-Mills partition function for $(\Sigma, \lambda_0 d\sigma)$ is then

$$Z_{\Sigma,\lambda_0 d\sigma} = \int_{G^{|E(\Gamma)|}} d\mu_{\Gamma,\lambda_0 d\sigma}. \tag{2.4}$$

Our lattice action is such that the partition function is independent of the choice of triangulation Γ. This is a simple consequence of the fact that the heat kernel obeys the convolution property

$$\int_G K_{t_1}(g_1 g^{-1}) K_{t_2}(gg_2)dg = K_{t_1+t_2}(g_1 g_2) \tag{2.5}$$

so that (2.4) is invariant under subdivision. See [43] for further details.

We can make formula (2.4) more explicit. A surface Σ of genus h can be represented as a $2h$-gon with sides appropriately identified. Applying the above formula using the (degenerate) triangulation whose only face is such a $2h$-gon, we obtain

$$Z_{\Sigma,\lambda_0 d\sigma} = \int_{G^{2h}} K_{\lambda_0|\Sigma|}(a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1}) \prod da_i \prod db_i.$$

In a quantum field theory, we are interested not only in the partition function but also in the expectation values of (gauge-invariant) observables. For continuum gauge theories, we have the Wilson loop observables $W_{f,\gamma}(A)$, which form a rich set of observables since they

form a dense collection of functions on the space of connections modulo gauge. This is a simple consequence of the following fact:

Lemma 2.1. *Let A_1 and A_2 be two connections on a principal G-bundle P over a connected base manifold M. Suppose that their holonomies about every loop based at some given point p agree. Then A_1 and A_2 are gauge-equivalent.*

PROOF. Given any path γ in M, let $P_i(\gamma)$ denote parallel transport from $\gamma(0)$ and $\gamma(1)$ using A_i. Our hypotheses imply that given any path γ joining p to any other point $q \in M$, the automorphism $P_i(\gamma)P_i(\gamma)^{-1}$ of P_q, the fiber of P over q, is independent of γ. Letting q vary, this yields for us a well-defined bundle automorphism. This automorphism is the desired gauge transformation intertwining A_1 and A_2. $\square$

In the lattice formulation, we would like to compute the expectation $\langle W_{f,\gamma} \rangle$ with respect to the lattice Yang-Mills measure induced by some triangulation Γ on Σ, with $W_{f,\gamma}$ suitably defined. The lattice formulation makes it clear how to express such an expectation value in terms of a closed formula. We assume γ is *regular*, namely, that it is a finite union of piecewise smoothly embedded curves. Since γ is regular, we can consider its image as an oriented finite graph $\hat{\gamma}$ on Σ (the maximal components on which γ is injective constitute the edges of $\hat{\Gamma}$). Embed $\hat{\gamma}$ in some triangulation Γ of Σ. Write $\hat{\gamma} = \pm e_k \cdots \pm e_1$ as a sequence of edges in the order that they occur in the parametrization of γ, with the $\pm$ according to whether the given orientation of e_i agrees with the one induced from γ. We then obtain $g_\gamma = g_{e_k}^{\pm 1} \cdots g_{e_1}^{\pm 1}$ as above.

Definition 2.2. We have

$$\langle W_{f,\gamma} \rangle = \frac{1}{Z_{\Sigma,\lambda_0 d\sigma}} \int_{G^{|E(\Gamma)|}} f(g_\gamma) d\mu_{\Gamma,\lambda_0 d\sigma}. \tag{2.6}$$

The subdivision invariance property of our lattice formulation implies that (2.6) does not depend on the choice of Γ containing $\hat{\gamma}$.

Strictly speaking, in the lattice formulation of gauge-theories, one chooses a very fine triangulation of Σ and Wilson loops with γ adapted to the triangulation. The above formula, which allows arbitrary regular γ (and then extends to arbitrary continuous γ) takes into account the continuum limit, since as the triangulation gets finer and finer, one can approximate arbitrary curves. The end result is that (2.6) provides us with an operational definition of the Yang-Mills measure. A more refined treatment [26, Definition 2.10.4] shows that such a Yang-Mills measure is a measure on $\mathcal{F}(L\Sigma, G)/\mathcal{G}$, the quotient space of functions $\mathcal{F}(L\Sigma, G)$ from $L\Sigma$ (the based loops on Σ) to G, modulo the action of the group $\mathcal{G}$ of gauge transformations. (Given $f \in \mathcal{F}(L\Sigma, G)$ and $g \in \mathcal{G}$, we have $(g \cdot f)(\gamma) = g(\gamma(0))f(\gamma)g(\gamma(0))^{-1}$.) Leaving out many details, such a measure is obtained by constructing a probability space in which one has a G-valued random variable H_γ, for every $\gamma \in L\Sigma$, defined as follows. Given γ (assumed to be regular without loss of generality), the law of H_γ is obtained by conditioning the Yang-Mills measure $d\mu_\Gamma$ on $G^{|E(\Gamma)|}$, where $\Gamma \supset \hat{\gamma}$; namely, one conditions on the element g_γ determined by γ. Such a law determines a G-valued process $\{H_\gamma\}_{\gamma \in L\Sigma}$ and thus a measure on $G^{L\Sigma} = \mathcal{F}(L\Sigma, G)$, which due to its invariance under $\mathcal{G}$, descends to a measure on $\mathcal{F}(L\Sigma, G)/\mathcal{G}$.

One can informally regard a measure on $\mathcal{F}(L\Sigma, G)/\mathcal{G}$ as a measure on the space $\mathcal{A}/\mathcal{G}$ of connections $\mathcal{A}$ modulo gauge transformations. Indeed, we have an inclusion $\mathcal{A}/\mathcal{G} \to \mathcal{F}(L\Sigma, G)/\mathcal{G}$, given by mapping a connection A to the function $\mathrm{hol}_{(\cdot)}(A)$, which takes a based loop γ to $\mathrm{hol}_\gamma(A)$. This makes sense for A sufficiently regular; if A is continuous say, then the resulting holonomy function will depend continuously on γ. Thus, $\mathcal{F}(L\Sigma, G)/\mathcal{G}$ represents "generalized connections". One can also interpret the Yang-Mills measure in terms of white-noise measures, with Wilson loop observables being given by stochastic parallel transport [17, 37].

We only mention these measure-theoretic interpretations so as to note that Yang-Mills theory in two dimensions has a rigorous construction in accords with the demands of constructive quantum field theory [22]. For our present purposes however, we are only concerned with the formula (2.6), which gives the exact expectation value of a Wilson loop observable.

We will be mainly concerned with the case $\Sigma = S^2$ for reasons explained in the introduction. To that end, let us specialize (2.6) to $\Sigma = S^2$ and the case of a simple closed curve, which we will analyze in Section 4.

Corollary 2.3. *Let γ be a simple closed curve on S^2, with R_1 and R_2 the connected components of $S^2 \setminus \gamma$. Then*

$$\langle W_{f,\gamma} \rangle = \frac{\int_G f(g) K_{\lambda_0|R_1|}(g) K_{\lambda_0|R_2|}(g) dg}{K_{\lambda_0|S^2|}(1)} \tag{2.7}$$

PROOF. The graph of γ consists of a single edge which we label by g. We now apply (2.6) with $\Gamma = \hat{\gamma}$ and make use of properties (2.2) and (2.5). $\square$

Remark 2.4. The above analysis carries over to $\mathbb{R}^2$ (equipped with the standard area form). For unbounded regions of $\mathbb{R}^2$, we use $K_\infty(g) \equiv 1$. Using the fact that $\int_G K_t(g) dg = 1$ for all t, the partition function $Z_{\mathbb{R}^2, \lambda_0 d^2 x}$ simply becomes unity. Since the area of $\mathbb{R}^2$ is not normalizable, the effective coupling constant for Yang-Mills theory on $\mathbb{R}^2$ is simply $\lambda = \lambda_0$. (We can think of λ as λ_0 times the area of the unit square, which is one.)

Remark 2.5. The Yang-Mills measure we describe in fact consists of an average of all possible topological bundle types over Σ [27]. In [27], a more refined construction associated to a graph $\mathcal{G}$ over Σ and a bundle type a Yang-Mills measure on a configuration space that covers $G^{E(\Gamma)}$. However, when $\lambda \to 0$, the dominant contribution to the Yang-Mills measure comes from the trivial bundle, since then one has the trivial connection in which all holonomies are trivial. (In other words, $K_t(g_{\partial F})$ is concentrated near $g_{\partial F} = 1$ for t small.) Nontrivial bundles will be asymptotically zero due to exponentially small factors arising from the topology (i.e. instantons). Thus, for our analysis at small coupling, it suffices to work with the averaged Yang-Mills measure described above.

We will compute the asymptotics of (2.7) as $\lambda = \lambda_0 |S^2| \to 0$ in Section 4 to give an explicit example of the relation between exact asymptotics and perturbation theory.

3. PERTURBATION THEORY

In the previous section, we discussed the full quantum Yang-Mills theory, in which one obtains expectation values for all possible Wilson loops $W_{f,\gamma}$ from a well-defined Yang-Mills

measure. This gives a complete, rigorous construction of the quantum Yang-Mills theory insofar as it provides a mathematical realization of the formal expression

$$\langle W_{f,\gamma} \rangle = \frac{\int dA \, W_{f,\gamma}(A) e^{-S_{YM}(A)}}{\int dA \, e^{-S_{YM}(A)}}, \tag{3.1}$$

which supposes the existence of a suitable Yang-Mills measure $dA \, e^{-S_{YM}(A)}$ on the space of connections (modulo gauge). Here, our basic field A is a connection on the trivial G-bundle over Σ so that the space of connections $\mathcal{A}$ can be identified with $\Omega^1(\Sigma; \mathfrak{g})$ and the group of gauge transformations can be identified with G-valued functions on Σ.

In this section, we discuss the *perturbative quantization* of Yang-Mills theory, whereby one computes expectation values not with regard to a true measure but through a perturbative expansion in Feynman diagrams about a minimal configuration of the classical action. In other words, we proceed by way of the standard paradigm of quantum field theory since its earliest inception: regard the right-hand side of (3.1) not as a number but as a notational device for generating a formal power series in the coupling constant λ.

The methods by which one generates such a formal power series, and the manner in which one establishes its resulting properties, are described with mixed approaches and differing degrees of rigor and generality in the physics and mathematics literature. For gauge theories, whatever approach one adopts, one ultimately needs to choose a gauge-fixing condition and show that the resulting outcome, i.e., the series expansion obtained from (3.1), is independent of the choice of gauge. For the case at hand, a degree of sophistication is required since we work on curved space, in which case the majority of treatments which quantize Yang-Mills theory on flat space do not readily apply. Indeed, flat space techniques such as working in momentum space and using dimensional regularization [13, 34] are not available to us.

We thus find it instructive to describe our quantization procedure from first principles, albeit in a succinct manner. That being so, *this section is written using both rigorous mathematics and the physically motivated ideas from which they are derived.* We trust that the reader finds this two-track narrative enlightening rather than confusing.

We have two main tasks. The first is to describe mathematically precisely how to generate the perturbative Feynman diagrammatic expansion of 2D Yang-Mills theory. Here, the simplest method is to use the standard Faddeev-Popov procedure. In doing so, one introduces a gauge-fixing procedure and corresponding ghosts fields. Moreover, one has to choose a suitable regularization and renormalization procedure. On curved space, the most natural regularization scheme is obtained from the use of the heat kernels. Using this scheme, we will show that in fact, 2D Yang-Mills theory in Coulomb gauge is finite. That is, no counterterms are needed to render the perturbative expectation values finite. Physically, this result can be anticipated from the fact that 2D Yang-Mills theory is superrenormalizable, so that if one assumes divergences are gauge-invariant (i.e. proportional to $|F_A|^2$) they will in fact be absent by power counting arguments[4]. We provide a proof of finiteness in Theorem 3.3.

The second task is to show that the formal power series we obtain, which gives a perturbative definition of the right-hand side of (3.1), is independent of the choice of gauge-fixing.

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[4]Such a naive assumption, however, is unjustified [1, 13].

This second step requires greater sophistication than that involved in the first step, whereby we use a blend of ideas from [4, 5, 12, 15] to establish gauge-invariance in Section 3.3. In broad strokes, we apply the Batalin Vilkovisky formalism in the form developed by [15], which provides a powerful algebraic framework by which to analyze the gauge-dependence of a quantization scheme. We find it less enticing to introduce this formalism from the start, since for the task of the defining the perturbative expansion and doing computations, the Faddeev-Popov procedure is more direct and illuminating. Thus, Section 3.3, while conceptually important and necessary, can be treated independently of the others.

3.1. **Faddeev-Popov quantization.** To quantize any classical gauge theory, we need to choose a suitable gauge-fixing procedure. This is because the path integrals in (3.1) should only be over the space of physically distinct configurations, i.e., those which are gauge-inequivalent.

Definition 3.1. A *compatible* Riemannian metric $g = g_{ij}$ is one for which the associated area form agrees with the given one $d\sigma$ on Σ occurring in the Yang-Mills action $\frac{1}{2e^2} \int_\Sigma \langle F_A \wedge *F_A \rangle$.

A compatible metric g_{ij} along with the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ yields for us an inner product $\int_\Sigma \langle \alpha \wedge *\beta \rangle$ on $\Omega^\bullet(\Sigma; \mathfrak{g})$ and a corresponding adjoint operator $d^* : \Omega^\bullet(\Sigma; \mathfrak{g}) \to \Omega^{\bullet-1}(\Sigma; \mathfrak{g})$ of the exterior derivative d. We consider the following *Coulomb gauge-fixing condition* $d^*A = 0$, and let

$$\mathcal{A}_C = \{A : d^*A = 0\}. \tag{3.2}$$

Since gauge transformations act via $A \mapsto gAg^{-1} + gdg^{-1}$, the above gauge-fixing condition eliminates all gauge degrees of freedom (with respect to based gauge transformations) in a neighborhood of the zero connection. We call g_{ij} a choice of *gauge-fixing metric*.

Our path integral

$$\frac{1}{Z} \int dA\, W_{f,\gamma}(A) e^{-S_{YM}(A)}$$

is to be replaced with the gauge-fixed path integral

$$\frac{1}{Z} \int_{\mathcal{A}_C} dA \det(d^*d_A) W_{f,\gamma}(A) e^{-S_{YM}(A)}. \tag{3.3}$$

The determinant factor $\det(d^*d_A)$ is the Faddeev-Popov determinant that weights gauge orbits appropriately[5]. This determinant can be evaluated via the introduction of anti-commuting fields, or ghosts. This is because there is a well-defined theory for fermionic integration in finite dimensions that produces this determinant factor (see Appendix B), and we can extrapolate from this an analogous procedure in the infinite dimensional case. We proceed as follows:

Introduce a pair of ghost fields $\omega, \bar{\omega}$, which each are $\mathfrak{g}$-valued functions on Σ. To keep them separate, we denote the space of ω and $\bar{\omega}$ by $\Omega^0(\Sigma; \mathfrak{g})$ and $\overline{\Omega}^0(\Sigma; \mathfrak{g})$, respectively. Define

$$\Omega_\perp^0 = \{\omega : \int_\Sigma \omega\, d\sigma = 0\}$$

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[5]For a rigorous treatment in the finite dimensional setting, see e.g. [31].

and similarly for $\overline{\Omega}_\perp^0$. Let

$$\mathfrak{C} = \Omega_\perp^0 \oplus \overline{\Omega}_\perp^0 \oplus \mathcal{A}_C,$$

consisting of the total space of gauge-fixed connections and ghosts that are orthogonal to constants. The latter condition is to eliminate the kernel of $d : \Omega^0(\Sigma; \mathfrak{g}) \to \Omega^1(\Sigma; \mathfrak{g})$, which arises because of infinitesimal gauge transformations acting trivially on the trivial connection.

We now replace (3.3) with

$$\frac{1}{Z} \int_{\mathfrak{C}} d\bar{\omega} d\omega dA \, W_{f,\gamma}(A) e^{-S(A,\bar{\omega},\omega)} \tag{3.4}$$

where[6]

$$S = \frac{1}{2e^2} \int \langle F_A \wedge *F_A \rangle + \frac{1}{e^2} \int \langle \bar{\omega}, d^* d_A \omega \rangle \, d\sigma.$$

The integration over $\omega, \bar{\omega}$ formally produces the determinant factor $\det(d^* d_A)$ via Lemma B.4.

It is with (3.4) that we can perform a Feynman diagrammatic expansion. This is done as follows. Group the extended Yang-Mills action into a quadratic kinetic part and the remaining higher order interaction part, which one regards as a perturbation of the former. Here, we use $F_A = dA + \frac{1}{2}[A, A]$ and $d_A = d + [A, \cdot]$. In this way, we have

$$e^{-S} = e^{-S_{kin}} e^I \tag{3.5}$$

where

$$S_{kin} = \frac{1}{2e^2} \int \langle A \wedge * d^* dA \rangle + \frac{1}{e^2} \int \langle \bar{\omega}, d^* d\omega \rangle \, d\sigma \tag{3.6}$$

$$I = -\frac{1}{2e^2} \int \langle [A, A] \wedge *dA \rangle - \frac{1}{8e^2} \int \langle [A, A] \wedge *[A, A] \rangle - \frac{1}{e^2} \int \langle \bar{\omega}, d^*[A, \omega] \rangle \, d\sigma \tag{3.7}$$

The next step is to write

$$\frac{1}{Z} \int_{\mathfrak{C}} d\bar{\omega} d\omega dA \, W_{f,\gamma}(A) e^{-S(A,\bar{\omega},\omega)} = \frac{1}{Z} \int_{\mathfrak{C}} d\bar{\omega} d\omega dA \, e^{-S_{kin}} W_{f,\gamma}(A) e^I \tag{3.8}$$

and then expand e^I as a formal series in powers of $\lambda_0 = e^2$. These terms, multiplied with $W_{f,\gamma}$, each give multilinear functionals on the space of fields. One then "integrates" each of these terms against the "Gaussian measure" $\frac{1}{Z} d\bar{\omega} d\omega dA e^{-S_{kin}}$ defined on $\mathfrak{C}$, thereby producing a formal power series in λ. In reality, what one is really doing is performing the algebraic operation given by Wick's Theorem. This operation is described in Appendix B. We describe how this generalizes to the quantum field theoretic setting at hand.

To invoke the appropriate analog of Lemma B.6, we need to describe the propagator P as well as the appropriate expansion of $W_{f,\gamma} e^I$ as a Taylor series (i.e. an infinite sum over polynomial functions). The Taylor expansion of e^I is automatic from the definition of the exponential function and we obtain a formal series in powers of λ. For $W_{f,\gamma}$, we obtain a

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[6]It is not necessary to multiply $\int \langle \bar{\omega}, d^* d_A \omega \rangle$ by $\frac{1}{e^2}$, but this ensures that all terms in the perturbative expansion are weighted equally in the coupling constant.

Taylor expansion via the representation of $\mathrm{hol}_\gamma(A)$ in terms of path ordered exponentials. Namely,

$$\mathrm{hol}_\gamma(A) = \mathcal{P}\exp\left(-\int_\gamma A\right)$$

$$= 1 + \sum_{n=1}^\infty (-1)^n \int_{1 \geq t_n \geq \cdots \geq t_1 \geq 0} A(t_1)\cdots A(t_n)$$

where $A(t) = A_\mu(\gamma(t))\gamma^\mu(t)$. Note that this presentation assumes we have chosen an embedding G into the group of unitary matrices $U(W)$ on a vector space W, so that elements of $\mathfrak{g} \subset \mathrm{End}(W)$ can be multiplied.

Without loss of generality, we can take $f = \mathrm{tr}_V$ to be trace in an irreducible representation $\rho : G \to \mathrm{End}(V)$, since the linear span of such functions is dense in the space of conjugation-invariant functions on G. This allows us to expand $W_{f,\gamma}(A)$ as a Taylor series in A:

$$W_{f,\gamma}(A) = \mathrm{tr}_V(1) + \sum_{n=1}^\infty (-1)^n \int_{1 \geq t_n \geq \cdots \geq t_1 \geq 0} \mathrm{tr}_V\Big(\rho(A(t_1))\cdots\rho(A(t_n))\Big). \qquad (3.9)$$

Here, we also write $\rho : \mathfrak{g} \to \mathrm{End}(V)$ to denote the induced Lie algebra homomorphism. The above representation uses the fact that parallel transport is equivariant with respect to group homomorphisms:

$$\rho(\mathrm{hol}_\gamma(A)) = \mathrm{hol}_\gamma(\rho(A)).$$

Altogether, this describes the expansion of the integrand $W_{f,\gamma}(A)e^I$ as a Taylor series in the field variables $A, \omega, \bar\omega$.

Next, the gauge-fixed path integral (3.4) determines for us a *propagator* P, which is the integral kernel of the Green's operator determined by S_{kin}. Specifically, we have the orthogonal decomposition

$$\mathcal{A} = \mathrm{im} * d \oplus \ker d$$

$$\Omega^0(\Sigma; \mathfrak{g}) = \Omega^0_\perp \oplus \mathbb{R}$$

$$\overline\Omega^0(\Sigma; \mathfrak{g}) = \overline\Omega^0_\perp(\Sigma) \oplus \mathbb{R}.$$

The decomposition for $\mathcal{A}$ depends on our compatible metric, while the other two depend only on $d\sigma$. The kinetic action S_{kin} is formed out of the Laplace-Beltrami operator $\Delta = \Delta_g$ restricted to $\mathfrak{C}$. The Green's operator we are interested in is the operator

$$P = \Delta^{-1}\big|_{\mathrm{im}\, *d \oplus \Omega^0_\perp \oplus \overline\Omega^0_\perp}$$

which extends to zero on the orthogonal complement. We have $P = P^{bos} \oplus P^{fer}$, according to the restriction of P to bosonic A and fermionic $\omega, \bar\omega$ fields.

The inner product on $\Omega^1(\Sigma; \mathfrak{g})$ allows us to identify the operator P^{bos} with its integral kernel

$$P^{bos}_{\mu,\nu}(x,y) \otimes \mathrm{id}_\mathfrak{g} \in \big(\mathrm{im}\, *d \otimes \mathrm{im}\, *d\big) \otimes \big(\mathfrak{g} \otimes \mathfrak{g}\big) \subset \big(\Omega^1(\Sigma) \otimes \Omega^1(\Sigma)\big) \otimes \big(\mathfrak{g} \otimes \mathfrak{g}\big). \qquad (3.10)$$

Here, we have separated the integral kernel of P^{bos} as the part $P^{bos}_{\mu,\nu}(x,y)$ which acts as Δ^{-1} on scalar-valued differential forms and the identity operator on $\mathfrak{g}$, which we view as

an element of $\mathfrak{g} \otimes \mathfrak{g}$ using the inner product on $\mathfrak{g}$. By abuse of notation, we also denote the integral kernel (3.10) simply by P^{bos}. It satisfies

$$\partial_{P^{bos}} A_\mu^a(x) A_\nu^b(x) = P_{\mu,\nu}^{bos}(x,y) \delta^{ab},$$

where a, b are $\mathfrak{g}$-indices with respect to an orthonormal basis of $\mathfrak{g}$. The fermionic propagator P^{fer} is the integral kernel of P restricted to $\Omega_\perp^0 \oplus \overline{\Omega}_\perp^0$ suitably interpreted. Namely, we have the pairing $\bar{\omega}, \omega \mapsto \int \langle \bar{\omega}, \omega \rangle \, d\sigma$ with which we can use to express the integral kernel of $\Delta^{-1} : \Omega_\perp^0 \to \Omega_\perp^0$ as an element of $\Omega^0 \otimes \overline{\Omega}^0$. Since Ω^0 and $\bar{\Omega}^0$ are fermionic fields, we pick up a minus sign when the order of ω and $\bar{\omega}$ is interchanged. Thus, P^{fer} is defined so as to satisfy

$$\partial_{P^{fer}} \omega^a(x) \bar{\omega}^b(y) = P^{fer}(x,y) \delta^{ab}$$
$$\partial_{P_{fer}} \bar{\omega}^a(x) \omega^b(y) = -P^{fer}(x,y) \delta^{ab}$$

This defines P^{fer} uniquely as an element of $\Lambda^2(\Omega_\perp^0 \oplus \overline{\Omega}_\perp^0)$. We have

$$P = P^{bos} + P^{fer} \tag{3.11}$$

is the (total) propagator.

We now suppose $\Sigma = S^2$. This way, $\operatorname{im} *d = \mathcal{A}_C$, and Δ restricted to $\mathcal{A}_C$ has no zero modes. This allows us to conclude that the Feynman diagrammatic expansion of (3.8), following Lemma B.6, is formally given by the expression[7]

$$\langle W_{f,\gamma} \rangle_C \text{ “}=\text{ ” } e^{\lambda_0 \partial_P} \left(W_{f,\gamma} e^I \right) \Big|_{conn,0}. \tag{3.12}$$

Here, the subscripts "$conn$" and "0" refer to the fact that we only wish to consider those Feynman diagrams which consist of a single component connected[8] to $W_{f,\gamma}$ and which have no external edges, respectively.

The formal definition (3.12) fails a priori because the resulting Feynman integrals we obtain are divergent. Thus, we need to choose a suitable regularization procedure, i.e. a way of mollifying the integral kernel P to a smooth one P_ϵ, $\epsilon > 0$, with $P_\epsilon \to P$ as $\epsilon \to 0$. Our regularization procedure is via the heat kernel method, which regulates P via

$$\int_\epsilon^\infty e^{-\Delta t} dt \Bigg|_{\operatorname{im} *d \oplus \Omega_\perp^0(\Sigma) \oplus \overline{\Omega}_\perp^0(\Sigma)}. \tag{3.13}$$

Note that as $\epsilon \to 0$, we recover $\Delta^{-1}|_{\mathfrak{c}}$, which is most easily seen by diagonalizing Δ and working on individual eigenspaces.

Let P_ϵ be the integral kernel of (3.13). It is smooth for all $\epsilon > 0$. We can replace P with P_ϵ in (3.12) and obtain a well-defined formal power series in λ. Having chosen a regularization procedure as above, we also need to perform renormalization, i.e. counterterms need to be

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[7]If there were zero modes, the definition of $\langle W_{f,\gamma} \rangle_{pert}$ should be modified to have a residual integration over these modes after performing the expansion (3.12).

[8]In path integral notation, the normalization factor $\frac{1}{Z}$ in (3.8) eliminates disconnected components of Feynman diagrams.

introduced. These are additional ϵ-dependent (and λ dependent) interactions I_ϵ^{CT} one adds to I. One is supposed to arrange the I_ϵ^{CT} so that

$$\lim_{\epsilon \to 0} e^{\lambda_0 \partial_{P_\epsilon}} \left(W_{f,\gamma} e^{I+I_\epsilon^{CT}} \right) \Big|_{conn,0}$$

exists as a formal power series in λ. (In general, additional counterterms may also be needed to renormalize observables.) A very nice feature of two-dimensional Yang-Mills theory is that in fact no counterterms are needed as $\epsilon \to 0$. This is proven in Theorem 3.3. Thus, we can form the following definition:

Definition 3.2. Fix a compatible metric. The perturbative expectation value of a Wilson loop $W_{f,\gamma}$ in the Coulomb gauge (3.2) is the formal series in $\lambda = \lambda_0 |\Sigma|$ defined by

$$\langle W_{f,\gamma} \rangle_C = \lim_{\epsilon \to 0} e^{\lambda_0 \partial_{P_\epsilon}} \left(W_{f,\gamma} e^I \right) \Big|_{conn,0}. \tag{3.14}$$

This yields for us a mathematically rigorous definition of the perturbative expectation value of a Wilson loop, with the preceding discussion revealing its physical origins. In the next section, we prove Theorem 3.3, thereby showing that the limit (3.14) exists. We then show that (3.14) is independent of the choice of gauge-fixing metric in Theorem 3.4.

3.2. **2D Yang-Mills is finite.** In this section, we prove the following result:

Theorem 3.3. *For a closed surface Σ, we have*

$$\lim_{\epsilon \to 0} e^{\lambda_0 \partial_{P_\epsilon}} \left(W_{f,\gamma} e^I \right) \Big|_{conn,0} \tag{3.15}$$

exists.

In two-dimensions, Yang-Mills theory is superrenormalizable, which means that there are only finitely many one-particle irreducible[9] diagrams which are potentially divergent. This is a simple inspection: the singularities of P are logarithmic, i.e. $P(x,y) \sim \log|x - y|$, so that the type of propagator insertions which lead to divergent integrals is highly constrained. Moreover, we only need consider those diagrams that have external edges (since these ultimately need to be connected to a Wilson loop observable).

It is straightforward to see that the only one-particle irreducible diagrams with external edges that are potentially divergent as $\epsilon \to 0$ are the ones in Figures I-III. Here we have omitted powers of λ_0 appearing in $\lambda_0 \partial_{P_\epsilon}$ and in I, since for these diagrams they cancel to give an overall factor of λ_0^0. Individually, these diagrams are divergent as $\epsilon \to 0$, but we are only interested in sums over Feynman diagrams as occurring in (3.15). What we will show is that the sum of these diagrams remains finite as $\epsilon \to 0$. The consequence is that the sum over *all* Feynman diagrams in (3.15) become finite as $\epsilon \to 0$, since the only possible divergences come from a subgraph consisting of the sum of diagrams I–III.

Observe that each Feynman diagram has two pieces: a Lie-algebraic part and an analytic part. This is because the propagator P factors into a differential-form part and the tensor

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[9]A Feynman diagram is one-particle irreducible if it cannot be disconnected by cutting an internal edge.
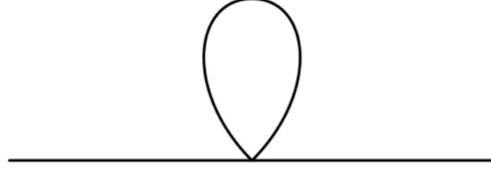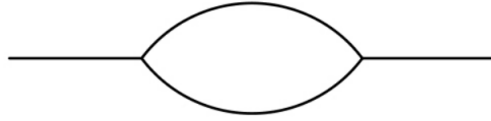
FIGURE I. Tadpole from 4-point vertex: $\partial_{\tilde{P}^{bos}_\epsilon}\left(-\frac{1}{8}\int\langle[A,A]\wedge *[A,A]\rangle\right)$

FIGURE II. Loop from 3-point vertex: $\frac{1}{2}\partial^2_{\tilde{P}^{bos}_\epsilon}\frac{1}{2}\left(-\frac{1}{2}\int\langle[A,A]\wedge *dA\rangle\right)^2$

FIGURE III. Ghost loop: $\frac{1}{2}\partial^2_{\tilde{P}^{fer}_\epsilon}\frac{1}{2}\left(-\int\langle\bar{\omega},d^*[A,\omega]\rangle\right)^2$

$\mathrm{id}_\mathfrak{g}\in\mathfrak{g}\otimes\mathfrak{g}$. Hence, the integrals arising from Feynman diagrams consist of Lie-algebraic contractions and differential-form contractions. For each of the above diagrams, we will compute each of these factors separately. (The individual Lie and analytic factors have an overall sign that depends on how one expresses Feynman diagrams as Wick contractions, but the product of these factors is always well-defined.) Here, a proper understanding of the signs and combinatorial factors attached to the different Feynman diagrams is absolutely essential, as they affect the sum leading to the cancellation of divergences. Moreover, observe that in analyzing the potential divergences of the above diagrams, we can use

$$\tilde{P}^{bos}_\epsilon=\int_\epsilon^\infty e^{-\Delta t}dt\bigg|_{\Omega^1(\Sigma;\mathfrak{g})} \tag{3.16}$$

instead of

$$P^{bos}_\epsilon=\Pi_{\mathrm{im}\,d^*}\int_\epsilon^\infty e^{-\Delta t}dt$$

for the bosonic propagator, where $\Pi_{\mathrm{im}\,d^*}$ is orthogonal projection onto $\mathrm{im}\,d^*$ using the gauge-fixing metric g. Indeed, the component of $\int_\epsilon^\infty e^{-\Delta t}dt$ that acts on $\ker d$ has no affect on Figure 2, since the propagator is contracted with the operator d and $d^2 = 0$. For Figure 1, the operator $\Pi_{\mathrm{im}\,d^*}$ in P_ϵ^{bos} is bounded and can be made to act on the two external legs; the internal loop is then simply given by $\tilde{P}_\epsilon^{bos}$. That the external legs acquire a projection onto $\mathrm{im}\,d^*$ is immaterial, since when these legs get contracted into other diagrams using the actual propagator P_ϵ, the projection $\mathrm{im}\,d^*$ has no effect.

Figure I. (tadpole on 4 point function):

The only Wick contractions $\partial_{P_\epsilon^{bos}}\left(-\frac{1}{8}\int \langle [A,A] \wedge *[A,A]\rangle\right)$ which are nonzero are those which pair A's from different Lie bracket terms (since P is proportional to $\mathrm{id}_{\mathfrak{g}}$). There are four such possible contractions.

Without loss of generality, regard the Wick contraction as contracting the second and third copy of A. In what follows, pick an orthonormal basis e_a for $\mathfrak{g}$ with respect to $\langle \cdot,\cdot\rangle$. We have the Lie algebra factor $\langle [e_a,e_c],[e_c,e_b]\rangle$, where e_a and e_b are Lie algebra factors coming from the external legs and we sum over repeated indices. Letting $C = \mathrm{Ad}(e_c)\mathrm{Ad}(e_c)$, we have

$$\langle [e_a,e_c],[e_c,e_b]\rangle = \langle [e_c,[e_c,e_a]],e_b\rangle$$
$$= C_{ab}$$

where C_{ab} is the ab matrix element of C. Thus we have

$$(I)_{Lie} = C_{ab} \tag{3.17}$$

Note that for G simple, $C_{ab} = -c_2(\mathrm{Ad}(\mathfrak{g}))\delta_{ab}$, where $c_2(\mathrm{Ad}(\mathfrak{g}))$ is the value of the quadratic Casimir in the adjoint-representation.

To compute the analytic factor of (I), write the external leg variable as $A = A^a e_a$, where A^a is a scalar-valued 1-form, which we can take to belong to $\mathrm{im}\,*d$. We can treat the propagator as a real-valued differential form on $\Sigma\times\Sigma$, since the Lie algebra factor has been accounted for. By abuse of notation, we use the same notation for scalar-valued counterparts in this setting. We have

$$(I)_{an} = 4\cdot -\frac{1}{8}\int_\Sigma A^a(x)\wedge (1\otimes *\otimes 1)(P_\epsilon(x,x)\wedge A^b(x)),$$

where 4 is the combinatorial factor arising from the number of Wick contractions. We have to understand the behavior of P near the diagonal. For $x,y\in\Sigma$ nearby, we can write $P(x,y)$ as a singular part $P^{sing}(x,y)$ plus a remaining smooth part. Since we can work with (3.16) instead of P, we consider

$$\tilde{P}^{bos,sing}(x,y) = -\frac{\log d(x,y)}{2\pi}\Phi_y(x) \tag{3.18}$$

where $d(x,y)$ is the Riemannian distance from x to y (with respect to g) and

$$\Phi_y(x)\in \mathrm{End}(T_y^*\Sigma, T_x^*\Sigma)$$
$$\Phi_y(y) = \mathrm{id}_{T_y^*\Sigma}$$

is smooth in x, y. The metric on Σ allows us to identify $\Phi_y(x) \in T_x^*\Sigma \otimes T_y^*\Sigma$. In particular, we can write

$$\Phi_x(x) = v_\mu(x) \otimes v_\mu(x)$$

for some auxiliary orthonormal frame $v_\mu(x)$ for $T_x^*\Sigma$.

For $\tilde{P}_\epsilon$, its restriction to the diagonal is a mollified version of (3.18), namely

$$\tilde{P}_\epsilon^{bos,sing}(x,x) = \frac{\log \epsilon^{-1/2}}{2\pi} v_\mu(x) \otimes v_\mu(x). \tag{3.19}$$

This immediately follows from the fact that $e^{-t\Delta} \sim \frac{1}{4\pi t} e^{-d(x,y)^2/4t}$ for x, y nearby as $t \to 0$. In other words, (3.19) shows that the heat kernel regulator (3.16) serves as a short distance cutoff, with the $1/2$–exponent occuring due to the quadratic relation between time and space. So

$$\begin{aligned}
(I)_{an} &= -\frac{1}{2} \frac{\log \epsilon^{-1/2}}{2\pi} \int_\Sigma A^a(x) \wedge v_\mu(x) \wedge *(v_\mu(x) \wedge A^b(x)) \\
&= \frac{\log \epsilon^{-1/2}}{4\pi} \int_\Sigma A^a \wedge *A^b.
\end{aligned}$$

Altogether, the Feynman integral corresponding to Figure I equals

$$(I) = (I)_{Lie}(I)_{an} \tag{3.20}$$

$$= \frac{\log \epsilon^{-1/2}}{4\pi} \int_\Sigma C_{ab} A^a \wedge *A^b. \tag{3.21}$$

Figure II. (one loop graph using 3 point vertices)

There are two possible types of Wick contractions that lead to divergences. For the first type, a propagator joins both copies of dA. For the second, dA belongs to separate Wick contractions. The number of possible Wick contractions in each case is 4. Call the resulting Feynman diagrams (II_1) and (II_2), respectively.

For (II_2), regard each dA of I_{bos} as being contracted with the rightmost copy of A in the $[A, A]$ in the other copy of I_{bos}. So we get the Lie-algebraic factor

$$\begin{aligned}
(II_2)_{Lie} &= \langle [e_a, e_c], e_d \rangle \langle [e_b, e_d], e_c \rangle \\
&= -\langle [e_a, e_c], e_d \rangle \langle e_d, [e_b, e_c] \rangle \\
&= -\langle [e_a, e_c], [e_b, e_c] \rangle \\
&= C_{ab}.
\end{aligned}$$

For the analytic factor, it is more notationally convenient to write P^{bos} and then regulate afterwards. We have two propagators, which we denote by P_1 and P_2.

$$(II_2)_{an} = 4 \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 \int_{\Sigma_x \times \Sigma_y} \left(A^a(x) \wedge P_1(x) \wedge *d_x P_2(x)\right)\left(A^b(y) \wedge P_2(y)) \wedge *d_y P_1(y)\right)$$

To extract the singular part, we restrict to a tubular neighborhood N of the diagonal. Let $\equiv$ denote equality up to terms that are finite as $\epsilon \to 0$. Using (3.19), we obtain

$$(II_2)_{\mathrm{an}} \equiv \frac{1}{4} \frac{1}{(2\pi)^2} \int_N \left[\left(A^a(x) \wedge v^{\mu_1}(x) \wedge * \left(\frac{(x-y)^{\nu_1}}{|x-y|^2} dx^{\nu_1}(x) \wedge v^{\mu_2}(x) \right) \right) \times \right.$$
$$\left. \left(A^b(y) \wedge v^{\mu_2}(y) \wedge * \left(\frac{(y-x)^{\nu_2}}{|x-y|^2} dy^{\nu_2} \wedge v^{\mu_1}(y) \right) \right) \right]$$

Only the $\mu_1 = \mu_2$ terms survive; else we get $\nu_1 \neq \nu_2$ in which case $(x-y)^1(x-y)^2$ is odd under interchange of x and y. Thus

$$(II_2)_{\mathrm{an}} \equiv -\frac{1}{4} \frac{1}{(2\pi)^2} \int_N (A^a(x) \wedge v^\mu(x)) \frac{|x-y|^2}{|x-y|^4} (A^b(y) \wedge v^\mu(y))$$

Integrating over y, we see that we get a logarithmic singularity at $y = x$. When we regulate, we obtain

$$(II_2)_{\mathrm{an}}^{sing} = -\frac{1}{4} \frac{\log \epsilon^{-1/2}}{2\pi} \int_\Sigma A^a \wedge * A^b.$$

For $(II_1)_{Lie}$, we have a Wick contraction of the two dA's; let the other Wick contraction be on the right-most A's. We get the Lie-algebraic factor:

$$(II_1)_{Lie} = \langle [e_a, e_c], e_d \rangle \langle [e_b, e_c], e_d \rangle$$
$$= -C_{ab}$$

For the analytic factor, for $\epsilon > 0$, the integral is convergent and one can integrate by parts to write $\int \langle [A, A] \wedge * dA \rangle = - \int \langle d * [A, A] \wedge A \rangle$. This way, we do not have two derivatives acting on any of the P_ϵ. A similar analysis above shows that the singular part of $(II_1)^{an}$ is given by

$$(II_1)_{an}^{sing} = -\frac{1}{4} \frac{\log \epsilon^{-1/2}}{2\pi} \int A \wedge * A.$$

Alternatively, one can also see that the Wick contraction of the two dA's converges to minus the delta function as $\epsilon \to 0$, in which case, the integral for $(II_1)_{an}^{sing}$ equals $-(I)_{an}$. Altogether, the Feynman integral corresponding to Figure II satisfies

$$(II)^{sing} = (II_1)_{Lie}(II_1)_{an}^{sing} + (II_2)_{Lie}(II_2)_{an}^{sing}$$
$$= -\frac{1}{4} \frac{\log \epsilon^{-1/2}}{2\pi} \int_\Sigma C_{ab} A^a \wedge * A^b + \frac{1}{4} \frac{\log \epsilon^{-1/2}}{2\pi} \int_\Sigma C_{ab} A^a \wedge * A^b$$
$$= 0.$$

Figure III. (ghost loop)

Here we have two ghost propagators. One contracts the outermost $\bar\omega$ and ω, the other contracts the inner ω with $\bar\omega$. The Lie-algebraic factor we get is

$$(III)_{Lie} = \langle e_c, [e_a, e_d] \rangle \langle e_d, [e_b, e_c] \rangle$$
$$= - \langle [e_a, e_c], [e_b, e_c] \rangle$$
$$= C_{ab}$$

For the analytic factor, we pick up an overall minus sign for the ghost loop. Indeed, the ordering of $\bar{\omega}$ and ω for the outermost contraction is the reverse of the second inner contraction, hence the first propagator picks up a minus sign since it is changes sign under interchange of these two fermionic fields. Thus

$$(III)_{an} = -\frac{1}{2} \int_{\Sigma} P_1^{fer}(x) d^*(A^a(x) P_2^{fer}(x)) d\sigma \int_{\Sigma} P_2^{fer}(y) d^*(A^b(y) P_1^{fer}(y)) d\sigma$$

$$= -\frac{1}{2} \int_{\Sigma} dP_1^{fer}(x) \wedge *A^a(x) P_2^{fer}(x) \int_{\Sigma} dP_2^{fer}(y) \wedge *A^b(y) P_1^{fer}(y).$$

Since $P_{fer}(x,y)$ has singular part $-\frac{1}{2\pi} \log d(x,y)$ near the diagonal, restricting to a tubular neighborhood N of the diagonal we get

$$(III)_{an} \equiv -\frac{1}{2} \frac{1}{(2\pi)^2} \int_N \left(\frac{|x-y|^{\mu}}{|x-y|^2} dx^{\mu} \wedge *A^a(x) \right) \frac{|x-y|^{\nu}}{|x-y|^2} dy^{\nu} \wedge *A^b(y)$$

$$= -\frac{1}{2} \frac{1}{(2\pi)^2} \int_N \frac{1}{2} \cdot \frac{|x-y|^2}{|x-y|^4} A^a \wedge *A^b.$$

As in case II, when we regulate, we conclude that

$$(III)_{an}^{sing} \equiv -\frac{1}{4\pi} \log \epsilon^{-1/2} \int_{\Sigma} A^a \wedge *A^b.$$

Thus, the Feynman integral corresponding to Figure III satisfies

$$(III)^{sing} = (III)_{Lie}(III)_{an}^{sing}$$

$$= -\frac{1}{4\pi} \log \epsilon^{-1/2} \int_{\Sigma} C_{ab} A^a \wedge *A^b.$$

Proof of Theorem 3.3: From the above computations, we find that

$$(I) + (II)^{sing} + (III)^{sing} = 0$$

Thus, the limit (3.15) is finite since all divergences cancel. $\square$

3.3. **Metric-independence of Coulomb gauge.** In this section, we prove the following theorem:

Theorem 3.4. *Let* $\Sigma = S^2$. *Then* $\langle W_{f,\gamma} \rangle_C$ *is independent of the choice of gauge-fixing metric.*

We prove this theorem using the Batalin-Vilkovisky (BV) formalism. The power of the BV formalism is that it captures the notion of gauge-invariance in an algebraic manner that is well-adapted for perturbative quantization. For additional background and insights regarding this formalism, we refer the reader to [15, 38, 42]. We will dive directly into the formalism, which is an adaptation of the approach of [15].

For our present purposes, a fundamental aspect of the BV formalism consists of being able to find a chain complex on which the propagator, viewed as an operator, becomes a chain homotopy between the identity map and the projection onto the zero modes of our theory. Let us unravel this rather involved statement. Consider the following chain complex

$$
\begin{array}{ccccc}
\text{degree}: & -1 & 0 & 1 & 2 \\
& \Omega^0(\Sigma, \mathfrak{g}) \xrightarrow{\ d\ } \Omega^1(\Sigma, \mathfrak{g}) \xrightarrow{\ d*d\ } \Omega^{1,\dagger}(\Sigma, \mathfrak{g}) \xrightarrow{\ d\ } \Omega^2(\Sigma, \mathfrak{g}) & & & \\
& X & A & A^\dagger & X^\dagger
\end{array}
\tag{3.22}
$$

consisting of the ghost field X, gauge field A, antifield $A^\dagger$, and antighost $X^\dagger$. (We use X instead of ω for our ghosts now, since in the BV formalism they appear in the action in a different form.) The space $\Omega^{1,\dagger}(\Sigma, \mathfrak{g})$ is just a separate copy of $\Omega^1(\Sigma; \mathfrak{g})$ to keep track of the field $A^\dagger$. We call this chain complex $\mathcal{E}$, which is our total space of (all) fields, and its components have degree listed as above.

Definition 3.5. A gauge-fixing operator $Q^\dagger : \mathcal{E} \to \mathcal{E}$ is an operator of degree -1 that satisfies (i) $(Q^\dagger)^2 = 0$; (ii) $[Q, Q^\dagger]$ is a generalized Laplace-type operator[10].

A gauge-fixing operator $Q^\dagger$ yields for us a Hodge-like decomposition $\mathcal{E} = \operatorname{im} Q \oplus \operatorname{im} Q^\dagger \oplus \ker[Q, Q^\dagger]$. More importantly, is allows us to define a Feynman diagrammatic expansion. In our situation, we are concerned with the following gauge-fixing operators:

Definition 3.6. Given a compatible metric $g = g_{ij}$ on S^2, define the gauge-fixing operator $Q_g^\dagger$ via

$$
\Omega^0(\Sigma, \mathfrak{g}) \xleftarrow{\ d^*\ } \Omega^1 \xleftarrow{-*\Pi_{\operatorname{im} d}} \Omega^{1,\dagger}(\Sigma, \mathfrak{g}) \xleftarrow{\ d^*\ } \Omega^2(\Sigma, \mathfrak{g})
\tag{3.23}
$$

Here, $\Pi_{\operatorname{im} d}$ denotes the orthogonal projection onto $\operatorname{im} d \subset \Omega^1(\Sigma, \mathfrak{g})$ with respect to g_{ij}.

Observe that $[Q, Q_g^\dagger] = \Delta_g$ acting in all degrees. On S^2,

$$
\mathcal{H} := \ker \Delta_g
$$

is spanned by the constant functions and the fixed area form $d\sigma$ on S^2. Thus, $\mathcal{H}$ is independent of the choice of compatible metric g. We can define the pseudoinverse $[Q, Q_g^\dagger]^{-1}$, which is zero on $\ker[Q, Q_g^\dagger]$ and the inverse of $[Q, Q_g^\dagger]$ on the orthogonal complement $\mathcal{E}_\perp$ to $\mathcal{H}$ (the complement $\mathcal{E}_\perp$ and the orthogonal projection onto it are also independent of compatible g).

A gauge-fixing operator allows us to construct a propagator. The propagator we get from $Q_g^\dagger$, which we call the BV propagator, differs from the propagator we obtained in the Faddeev-Popov procedure in the previous section. However, BV quantization leads to the same set of Feynman integrals as Faddeev-Popov quantization, as we will see shortly.

Definition 3.7. Given the gauge-fixing operator $Q_g^\dagger$, we obtain the corresponding *BV propagator* $\hat{P}_g = Q_g^\dagger [Q, Q_g^\dagger]^{-1}$. It is a degree -1 operator on $\mathcal{E}$ which satisfies

$$
[Q, \hat{P}_g] = \operatorname{id} - \Pi
\tag{3.24}
$$

where Π is the orthogonal projection onto $\mathcal{H}$.

---

[10]This means the operator is of the form $\nabla^* \nabla + F$, where ∇ is a covariant derivative, ∇^* its adjoint, and F a bundle endomorphism. Also, in what follows we must interpret the commutator $[\cdot, \cdot]$ in the graded sense as explained in the appendix.

Equation (3.24) is the statement that $\hat{P}_g$ is a chain homotopy from id and Π. The key significance of this fact is the following. The right-hand side of (3.24) is g-independent. Thus,

$$[Q, d_{met}\hat{P}_g] = 0 \tag{3.25}$$

where d_{met} is the exterior derivative on the space of compatible metrics on S^2 (such a space is a smooth, connected subvariety inside the space of all metrics)[11].

Equation (3.25) is the statement that $d_{met}\hat{P}_g$ is closed as an element $\mathrm{Hom}_\mathbb{R}(\mathcal{E}_\perp)$, the space of $\mathbb{R}$ linear maps on $\mathcal{E}_\perp$, endowed with the differential $[Q, \cdot]$ naturally induced from $\mathcal{E}$. On the other hand, since Q is acyclic on $\mathcal{E}^\perp$, it follows that $[Q, \cdot]$ is acyclic on $\mathrm{Hom}_\mathbb{R}(\mathcal{E}_\perp)$. Thus, we have

$$d_{met}\hat{P}_g \in \mathrm{im}\,[Q, \cdot]. \tag{3.26}$$

This is the *key identity* which allows us to establish gauge-invariance. Observe that Q arises from infinitesimal gauge-transformations (and the linearized equations of motion). Thus, one can interpret equation (3.26) as stating that the propagator $\hat{P}_g$, under changes of the metric g, changes by gauge degrees of freedom. Because the underlying classical theory is gauge-invariant, this leads to gauge-invariance of the quantum theory.

The remainder of this section makes the above remarks precise. We need to do the following:

(i) Convert the BV propagator $\hat{P}_g$, defined as an operator, into an integral kernel P_g so as to be placed on the edges of Feynman diagrams;

(ii) Describe the BV action so as to obtain the vertices to be used in Feynman diagrams;

(iii) Use (3.26) and the underlying gauge invariance of classical Yang-Mills theory and Wilson loop observables to establish gauge-invariance of Wilson loop expectation values. (This exploits the fact that no counterterms, in particular those that might have spoiled gauge-invariance, are needed for quantization.)

We have streamlined our approach in this manner because it then explains what would otherwise be many mysterious sign rules in what follows. All such choices of signs can be viewed as being carefully crafted so as to ensure (i)–(iii) above hold. In what follows, we carry out steps (i)–(iii) in a somewhat abstract, but coordinate-independent manner. Explicit computations to help make our approach more explicit are carried out in Remark 3.16.

Step (i):

To convert an operator to an integral kernel, we need a suitable pairing that induces a convolution operator. In other words, given a linear operator $\hat{K} : \mathcal{E} \to \mathcal{E}$ and a bilinear pairing $\langle \cdot, \cdot \rangle_\mathbb{R} : \mathcal{E} \otimes \mathcal{E} \to \mathbb{R}$, we want to express $\hat{K}$ as an element[12] $K \in \mathcal{E} \otimes \mathcal{E}$ in the following way. For $a \in \mathcal{E}$, we want $\hat{K}(a)$ to be given by applying $1 \otimes \langle \cdot, \cdot \rangle_\mathbb{R}$, with the appropriate signs to $K \otimes a \in \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E}$, which is to say that K is the integral kernel of $\hat{K}$ with respect to our chosen bilinear pairing.

---

[11]For those hesitant with working on infinite dimensional manifolds, one can always restrict to finite-dimensional families of metrics. Since all such families are compatible under union and restriction, taking a direct limit recovers the full infinite-dimensional family of compatible metrics.

[12]In what follows, all tensor products are completed, see footnote 17.

The pairing $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ and correct sign rule is determined by the following constraint. We want

$$k([Q,\hat{K}]) = Qk(\hat{K})$$

where k is the map taking an operator to its integral kernel. Here Q acts on $\mathcal{E}\otimes\mathcal{E}$ in the natural way as a derivation: $Q(a\otimes b) = Qa\otimes b + (-1)^{|a|}a\otimes Qb$.

Definition 3.8. Define the pairing $\langle\cdot,\cdot\rangle_{BV} : \mathcal{E}\otimes\mathcal{E} \to \Omega^2(\Sigma)$ by

$$\left\langle A, A^{\dagger}\right\rangle_{BV} = \left\langle A\wedge A^{\dagger}\right\rangle$$
$$\left\langle A^{\dagger}, A\right\rangle_{BV} = \left\langle A^{\dagger}\wedge A\right\rangle$$
$$\left\langle X, X^{\dagger}\right\rangle_{BV} = -\left\langle X\wedge X^{\dagger}\right\rangle$$
$$\left\langle X^{\dagger}, X\right\rangle_{BV} = \left\langle X^{\dagger}\wedge X\right\rangle$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathfrak{g}$. We then obtain the pairing

$$\langle\cdot,\cdot\rangle_{\mathbb{R}} = \int_{\Sigma}\langle\cdot,\cdot\rangle_{BV} : \mathcal{E}\otimes\mathcal{E} \to \mathbb{R} \tag{3.27}$$

Both these pairings are of degree -1, i.e. $\langle a,b\rangle_{\mathbb{R}}\neq 0$ only for $|a| + |b| = 1$, and are skew-symmetric.

In particular, $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is an odd symplectic form, i.e., it is skew-symmetric, nondegenerate, and of odd degree. From $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, we obtain a corresponding convolution operator on $\mathcal{E}$:

Definition 3.9. Given $K\in\mathcal{E}\otimes\mathcal{E}$, define the convolution operator $K* : \mathcal{E}\to\mathcal{E}$ as follows. On simple tensors $K = K_1\otimes K_2\in\mathcal{E}\otimes\mathcal{E}$, we have

$$K * a = (-1)^{|K_2|}K_1\langle K_2, a\rangle_{\mathbb{R}} \tag{3.28}$$

This determines $*$ on general K from bilinearity and completion. We say that K is the *BV integral kernel* of $K*$.

Note that if $|K| = d$, then $K*$ has degree $d + 1$. The sign rules in the definition of the BV pairing and in the definition (3.28) are carefully chosen so that the following are true:

Lemma 3.10. *We have that*

(i) *Q is skew-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, i.e.*

$$\langle a, Qb\rangle_{\mathbb{R}} = -(-1)^{|a|}\langle Qa, b\rangle_{\mathbb{R}}$$

(ii) *For all $K\in\mathcal{E}\otimes\mathcal{E}$, we have $[Q, K*] = [QK, *]$.*

PROOF. These are both straightforward computations. We check (ii):

$$[Q, K*](a) = Q(-1)^{|K_2|}K_1\langle K_2, a\rangle_{\mathbb{R}} - (-1)^{|K|+1}(-1)^{|K_2|}K_1\langle K_2, Qa\rangle_{\mathbb{R}}$$
$$(QK) * a = (-1)^{|K_2|}QK_1\langle K_2, a\rangle_{\mathbb{R}} + (-1)^{|K_1|}(-1)^{|K_2|+1}K_1\langle QK_2, a\rangle_{\mathbb{R}}$$

We use the skew-adjointness property (i) to equate the above two lines. $\square$

Given our propagator operator $\hat{P}_g$, we obtain the corresponding propagator integral kernel $P_g \in \mathcal{E} \otimes \mathcal{E}$ via

$$\hat{P}_g = P_g * . \tag{3.29}$$

The regulated propagator $P_{g,\epsilon}$ is obtained from regulating $[Q, Q_g^\dagger]^{-1}$ in $\hat{P}_g$:

$$\hat{P}_{g,\epsilon} = Q_g^\dagger \int_\epsilon^\infty e^{-[Q,Q_g^\dagger]t} dt \tag{3.30}$$

$$\hat{P}_{g,\epsilon} = P_{g,\epsilon} * . \tag{3.31}$$

Step (ii):

In the BV formalism, we have an extended BV action S consisting of the ordinary (bosonic) action S_{bos}, a ghost action S_{gh} which encodes infinitesimal gauge-symmetries, and a Chevalley-Eilenberg action S_{CE}, which accounts for the Lie-algebraic structure of the infinitesimal gauge-transformations. These are functions on the space $\mathcal{E}$ in the sense of Definition A.1, i.e., they are elements of $\mathrm{Sym}(\mathcal{E}^*)$. Explicitly,

$$S = S_{bos} + S_{gh} + S_{CE}$$

$$S_{bos}(A) = S_{YM}(A)$$

$$S_{gh}(A^\dagger, A, X) = \frac{1}{e^2} \left\langle A^\dagger, -d_A X \right\rangle_{\mathbb{R}}$$

$$S_{CE}(X^\dagger, X_1, X_2) = -\frac{1}{e^2} \left\langle X^\dagger, [X_1, X_2] \right\rangle_{\mathbb{R}}. \tag{3.32}$$

The quadratic part of S yields a kinetic term

$$S_{kin} = \frac{1}{2e^2} \int \langle A, d * dA \rangle - \frac{1}{e^2} \int \left\langle A^\dagger, dX \right\rangle \tag{3.33}$$

while the negative of the cubic and quartic parts of S yield for us the interaction terms:

$$I = -\frac{1}{2e^2} \int \langle [A, A] \wedge *dA \rangle - \frac{1}{8e^2} \int \langle [A, A] \wedge *[A, A] \rangle + \frac{1}{e^2} \int \left\langle A^\dagger \wedge [A, X] \right\rangle - S_{CE}. \tag{3.34}$$

Define I_{bos}, I_{gh}, and I_{CE} to be the first two, the third, and last terms of I, respectively.

The basis for the extended action is as follows. The BV pairing induces a BV bracket, which allows us to convert action functionals to vector fields. Conceptually, it is the (odd) Poisson bracket corresponding the BV pairing.

Definition 3.11. Let F be a local functional, i.e., one given by the integral of a polydifferential function of the fields. Then there is a unique local vector field $\delta^{BV} F$ such that

$$\partial_v F = (-1)^{|v|} \int_\Sigma \left\langle v, \delta^{BV} F \right\rangle_{BV}$$

for all $v \in \mathcal{E}$. The *BV bracket* $\{F, G\}$ between a local functional F and an arbitrary functional G is given by

$$\{F, G\} = \partial_{\delta^{BV} F} G.$$

where one must interpret $\partial_{\delta^{BV} F}$ in the sense of (A.3).

The BV action, which has degree zero, satisfies the following *master equation*[13]:

$$\{S, S\} = 0. \tag{3.35}$$

This equation can be written as

$$-QI + \frac{1}{2}\{I, I\} = 0, \tag{3.36}$$

since $e^2\{S_{kin}, \cdot\} = Q$ regarded as a derivation on the space $\mathrm{Sym}(\mathcal{E}^*)$ (the action of Q on $\mathcal{E}^*$ itself is the one induced from Q on $\mathcal{E}$ via pullback). In turn, (3.36) decomposes into three separate equations,

$$-Q_0 I_{gh} - Q_{-1} I_{bos} + \{I_{gh}, I_{bos}\} = 0 \tag{3.37}$$

$$-Q_{-1} I_{gh} - Q_1 I_{CE} + \frac{1}{2}\{I_{gh}, I_{gh}\} + \{I_{CE}, I_{gh}\} = 0 \tag{3.38}$$

$$\{I_{CE}, I_{CE}\} = 0. \tag{3.39}$$

which depend on one, two, and three ghost fields, respectively. Here Q_i corresponds to the part of Q which maps $\mathcal{E}_i$ to $\mathcal{E}_{i+1}$. Equation (3.37) expresses gauge-invariance of the classical action S_{bos}, (3.38) expresses that infinitesimal gauge transformations act as a Lie algebra, and (3.39) expresses the Jacobi identity.

Thus the master equation encodes symmetries and their algebraic consistency relations into a single equation. Expressing all such relations in a compact manner facilitates the analysis of symmetries, and in particular gauge-invariance, when quantizing. Next, we describe the Feynman diagrammatic expansion in the BV formalism, which involves applying Wick's Theorem to the BV propagator and the above BV interaction. It yields the same expansion as the Faddeev-Popov prcoedure, as the following lemma shows:

Lemma 3.12. *Fix a compatible metric. Consider the Faddeev-Popov propagator (3.11) and interactions (3.7) and the Batalin-Vilkovisky propagator (3.29) and interactions (3.34) which for notational clarity we denote by here by P^{FP}, I^{FP}, P^{BV}, and I^{BV}, respectively. Then*

$$\lim_{\epsilon \to 0} e^{\lambda_0 \partial_{P^{BV}_\epsilon}} e^{I^{BV}}\Big|_0 = \lim_{\epsilon \to 0} e^{\lambda_0 \partial_{P^{FP}_\epsilon}} e^{I^{FP}}\Big|_0 \tag{3.40}$$

As a consequence, we have

$$\lim_{\epsilon \to 0} e^{\lambda_0 \partial_{P^{BV}_\epsilon}} W_{f,\gamma} e^{I^{BV}}\Big|_{conn,0} = \lim_{\epsilon \to 0} e^{\lambda_0 \partial_{P^{FP}_\epsilon}} W_{f,\gamma} e^{I^{FP}}\Big|_{conn,0} \tag{3.41}$$

PROOF. By direct inspection, the bosonic parts of the propagators and interactions coincide. So the only remaining issue consists in comparing the fermionic propagators and the interactions that have fermions. We can ignore I_{CE} from the BV interactions because the propagator has no $X^\dagger$ component, so that upon setting external leg variables equal to zero, those that depend on $X^\dagger$ are annihilated. One can show that the fermionic propagators are related as follows. Note that

$$P^{BV,\mathrm{fer}} \in \left(\Omega^0(\Sigma; \mathfrak{g}) \otimes \Omega^{1,\dagger}(\Sigma; \mathfrak{g})\right) \oplus \left(\Omega^{1,\dagger}(\Sigma; \mathfrak{g}) \otimes \Omega^0(\Sigma; \mathfrak{g})\right)$$

---

[13]Note that because $\{\cdot, \cdot\}$ is an odd bracket, it satisfies $\{F, G\} = (-1)^{|F||G|}\{G, F\}$ for F and G local so that the master equation is not vacuous.

corresponding to $\hat{P}_g : \Omega^1(\Sigma; \mathfrak{g}) \to \Omega^0(X; \mathfrak{g})$ and $\hat{P}_g : \Omega^2(\Sigma; \mathfrak{g}) \to \Omega^{1,\dagger}(\Sigma; \mathfrak{g})$, respectively. On the other hand, we know that

$$P^{FP,\text{fer}} \in \left(\Omega^0(\Sigma; \mathfrak{g}) \otimes \overline{\Omega}^0(\Sigma; \mathfrak{g}) \right) \oplus \left(\overline{\Omega}^0(\Sigma; \mathfrak{g}) \otimes \Omega^0(\Sigma; \mathfrak{g}) \right).$$

Next, observe that $P^{BV,\text{fer}}$ is given by $(1 \otimes *d)$ and $(*d \otimes 1)$ applied to the two corresponding components of $P^{FP,\text{fer}}$. This follows, for instance, from the computation

$$\int_{\Sigma_y} (1_x \otimes *d_y) P^{FP,\text{fer}}(x, y) \wedge A(y) = - \int_{\Sigma_y} P^{FP,\text{fer}}(x, y) \wedge *d^* A(y)$$
$$= -\Delta^{-1} d^* A$$
$$= -\hat{P}^{BV} A.$$

(The implicit $\mathfrak{g}$-inner product in the Σ_y integral has been suppressed from the above notation.) It follows that $(1 \otimes *d)$ applied to the $\Omega^0 \boxtimes \overline{\Omega}^0$ component of $P^{FP,fer}$ is equal to the $\Omega^0 \boxtimes \Omega^{1,\dagger}$ component of $P^{BV,fer}$ due to the sign rule (3.28). On the other hand, if we identify $*d\bar{\omega} \in \operatorname{im} Q_g^\dagger$ with $A^\dagger$, then

$$- \int \langle \bar{\omega}, d^*[A, \omega] \rangle \, d\sigma = \int \langle *d\bar{\omega}, [A, \omega] \rangle$$
$$\longrightarrow \int \left\langle A^\dagger, [A, \omega] \right\rangle.$$

So I_{gh}^{FP} (the third term of (3.7)) and I_{gh}^{BV} are equal under this correspondence. One can now easily see that (3.40) and (3.41) hold. $\square$

Step (iii):

We now turn to the heart of the proof. We have a few algebraic preliminaries to establish:

Lemma 3.13. *Let $v \in \mathcal{E}$. For any linear operator D on $\mathcal{E}$, which acts as a derivation on* $\operatorname{Sym}(\mathcal{E})$ *and dually as a derivation on* $\operatorname{Sym}(\mathcal{E}^*)$, *we have*

(i) $\partial_{Dv} = [\partial_v, D]$
(ii) $\partial_{DK} = [\partial_K, D]$

as operators on $\operatorname{Sym}(\mathcal{E}^*)$.

PROOF. (i) Both ∂_{Dv} and $[\partial_v, D]$ are derivations of degree $(-1)^{|Dv|}$, so it suffices to check $\partial_{Dv} = [\partial_v, D]$ on $\mathcal{E}^*$. The statement is then automatic.

(ii) Without loss of generality, let $K = K_1 \otimes K_2$. Using (i), then

$$
\begin{aligned}
\partial_{DK} &= \partial_{(DK_1 \otimes K_2 + (-1)^{|D||K_1|} K_1 \otimes DK_2)} \\
&= \frac{1}{2} \left(\partial_{K_2} \partial_{DK_1} + (-1)^{|D||K_1|} \partial_{DK_2} \partial_{K_1} \right) \\
&= \frac{1}{2} \left(\partial_{K_2} [\partial_{K_1}, D] + (-1)^{|D||K_1|} [\partial_{K_2}, D] \partial_{K_1} \right) \\
&= \frac{1}{2} [\partial_{K_2} \partial_{K_1}, D] \\
&= [\partial_K, D]. \ \square
\end{aligned}
$$

Let K_t denote the BV integral kernel of $e^{-t\Delta}$, $t \geq 0$.

Definition 3.14. Define the BV Laplacian Δ^{BV} to be the "divergence operator"

$$
\Delta^{BV} := \partial_{K_0}
$$

Note that since K_0, being the integral kernel of the identity operator, is a δ-function, $\Delta^{BV}(F)$ is not well-defined for arbitrary F. Moreover, K_∞ is the BV integral kernel of Π.

Lemma 3.15. *If at least one of F or G is a local action functional, then*

$$
\{F, G\} = \Delta^{BV}(FG) - \Delta^{BV}(F)G - (-1)^{|F|} F \Delta^{BV}(G). \tag{3.42}
$$

PROOF. This is a straightforward computation by noting that $\{F, G\}$ is the part of $\partial_{K_0}(FG)$ in which the K_0 contraction joins F to G. This is what the right-hand side of (3.42) expresses, since it subtracts from $\Delta^{BV}(FG)$ those contractions that involve only F or G alone. $\square$

Remark 3.16. Here, we perform some explicit computations, albeit in a condensed and somewhat formal way, to illustrate how our sign conventions work (which we hid behind a veil of rather abstract constructions). First, beginning with a finite-dimensional setting, suppose we have a symplectic basis ∂_{x^i} and ∂_{ξ^j} of the BV bracket, i.e.

$$
\left\langle \partial_{x^i}, \partial_{\xi^j} \right\rangle_{BV} = \delta_{ij} = -\left\langle \partial_{\xi^j}, \partial_{x^i} \right\rangle_{BV},
$$

with ∂_{x^i} even and ∂_{ξ^j} odd. Then the identity operator has integral kernel

$$
K_0 = \partial_{x^i} \otimes \partial_{\xi^i} + \partial_{\xi^i} \otimes \partial_{x^i}.
$$

The divergence operator is then

$$
\Delta^{BV} = \partial_{K_0} = \partial_{x^i} \partial_{\xi^i}
$$

and the BV bracket is

$$
\{F, G\} = \partial_{\xi^i} F \partial_{x^i} G + (-1)^{|F|} \partial_{x^i} F \partial_{\xi^i} G.
$$

In the Yang-Mills setting, we have the even tangent vectors $\frac{\delta}{\delta A(x)}, \frac{\delta}{\delta X^\dagger(x)}$ and the odd tangent vectors $\frac{\delta}{\delta A^\dagger(x)}, \frac{\delta}{\delta X(x)}, x \in \Sigma$, where we have suppressed Lie-algebraic and differential-form indices. So

$$\Delta^{BV} = \int_\Sigma d^2x \left(\frac{\delta}{\delta A(x)} \frac{\delta}{\delta A^\dagger(x)} + \frac{\delta}{\delta X^\dagger(x)} \frac{\delta}{\delta X(x)} \right)$$

and using[14]

$$\left\langle \frac{\delta}{\delta A(x)}, \frac{\delta}{\delta A^\dagger(y)} \right\rangle_{BV} = \delta^{(2)}(x-y) = -\left\langle \frac{\delta}{\delta A^\dagger(x)}, \frac{\delta}{\delta A(y)} \right\rangle_{BV} \tag{3.43}$$

$$\left\langle \frac{\delta}{\delta X^\dagger(x)}, \frac{\delta}{\delta X(y)} \right\rangle_{BV} = \delta^{(2)}(x-y) = -\left\langle \frac{\delta}{\delta X(x)}, \frac{\delta}{\delta X^\dagger(y)} \right\rangle_{BV} \tag{3.44}$$

one can check that

$$e^2\{S_{kin}, \cdot\} = Q$$
$$= \int d^2x \left(dX(x) \frac{\delta}{\delta A(x)} + d * dA(x) \frac{\delta}{\delta A^+(x)} + dA^\dagger \frac{\delta}{\delta X^\dagger(x)} \right)$$

and that the master equation holds:

$$(3.37) = -\langle d * dA, [A, X] \rangle_\mathbb{R} + \left\langle dX, \frac{\delta I_{bos}}{\delta A} \right\rangle_\mathbb{R} + \left\langle [A, X], \frac{\delta I_{bos}}{\delta A} \right\rangle_\mathbb{R}$$
$$= \left\langle d_A X, \frac{\delta S_{bos}}{\delta A} \right\rangle_\mathbb{R}$$
$$= 0$$

$$(3.38) = -\left(dX_1 \frac{\delta}{\delta A} \llcorner \left\langle A^\dagger, [\cdot, X_2] \right\rangle_\mathbb{R} + \leftrightarrow \right) + -\left\langle dA^\dagger, [X_1, X_2] \right\rangle_\mathbb{R}$$
$$\quad - [A, X_1] \frac{\delta}{\delta A} \llcorner \left\langle A^\dagger, [\cdot, X_2] \right\rangle_\mathbb{R} + [X_1, X_2] \frac{\delta}{\delta X} \llcorner \left\langle A^\dagger, [A, \cdot] \right\rangle_\mathbb{R}$$
$$= \left(\left\langle A^\dagger, [d_A X_1, X_2] \right\rangle_\mathbb{R} + \leftrightarrow \right) - \left\langle A^\dagger, d[X_1, X_2] \right\rangle_\mathbb{R} - \left\langle A^\dagger, [A, [X_1, X_2]] \right\rangle_\mathbb{R}$$
$$= 0$$

$$(3.39) = 2[X_1, X_2] \frac{\delta}{\delta X} \llcorner \left\langle X^\dagger, [\cdot, X_3] \right\rangle_\mathbb{R} + \leftrightarrow$$
$$= 2 \left\langle X^\dagger, [[X_1, X_2], X_3] \right\rangle_\mathbb{R} + \leftrightarrow$$
$$= 0.$$

In the above, $\leftrightarrow$ means we symmetrize (in the graded sense) over the odd X's and $\llcorner$ is the contraction of a vector field with a one-form (and the appropriate sign rules must be used).

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[14]The minus sign occuring when we switch A and $A^\dagger$ in (3.43) arises from the skew-symmetry of the wedge pairing on 1-forms, not from a minus sign occurring in the definition of the BV bracket, as is the case for the right-hand side of (3.44). This minus sign later reappears as a minus sign in the third term of the identity involving (3.38) worked out below. These minus signs become more transparent if we restore the 1-form indices we have suppressed.

For (3.38), we obtain the vanishing of (3.38) from d_A being a derivation with respect to $[\cdot, \cdot]$. The vanishing of (3.39) follows from the Jacobi identity.

Proof of Theorem 3.4: Let $O = W_{f,\gamma}$ be a Wilson loop observable. We want to compute the variation of $\langle O \rangle_C$ with respect to variations in the choice of compatible metric g. We know that

$$d_{met} P_g = -Q P_g'$$

for some $P_g' \in \mathcal{E}_\perp \otimes \mathcal{E}_\perp$ via (3.26) and Lemma 3.13(ii). Since P_g has degree zero and $[Q, \cdot]$ has degree one, P_g' has degree -1. However, if we regard P_g' as having differential-form degree one in the metric directions, then this total degree of P_g' is zero.

It is convenient to combine P_g and P_g' into a master propagator

$$\tilde{P}_g = P_g + P_g'.$$

Let $=_1$ denote equality between components of degree one in the metric directions.

Then we have

$$d_{met} \langle O \rangle_C =_1 d_{met} \left(e^{\lambda_0 \partial_{\tilde{P}_g}} O e^I \right) \Big|_{conn,0}$$

$$=_1 \lambda_0 \left(e^{\lambda_0 \partial_{\tilde{P}_g}} \partial_{d_{met}\tilde{P}_g} O e^I \right) \Big|_{conn,0}. \tag{3.45}$$

where we are implicitly taking the $\epsilon \to 0$ limit of regulated propagators. We have

$$d_{met} \tilde{P}_g =_1 d_{met} P_g$$

$$=_1 -Q P_g'$$

$$= -Q \tilde{P}_g + (\lim_{\epsilon \to 0} K_\epsilon - K_\infty).$$

In the last line, we used the regulated version of (3.24). Consider the regulated BV divergence operator

$$\tilde{\Delta}^{BV} := \lim_{\epsilon \to 0} \partial_{K_\epsilon}.$$

We have

$$(3.45) =_1 \lambda_0 \left(e^{\lambda_0 \partial_{\tilde{P}_g}} (-\partial_{Q P_g} + \lim_{\epsilon \to 0} \partial_{K_\epsilon} - \partial_{K_\infty}) O e^I \right) \Big|_{conn,0}$$

$$=_1 \lambda_0 \left(e^{\lambda_0 \partial_{\tilde{P}_g}} (-Q + \tilde{\Delta}^{BV} - \partial_{K_\infty}) O e^I \right) \Big|_{conn,0}$$

$$=_1 \lambda_0 \left(e^{\lambda_0 \partial_{\tilde{P}_g}} \left((-QO + \{I, O\}) + \tilde{\Delta}^{BV} O + (-QI + \frac{1}{2}\{I, I\} + \tilde{\Delta}^{BV} I) \right) e^I \right) \Big|_{conn,0}$$

$$- \lambda_0 \left(e^{\lambda_0 \partial_{\tilde{P}_g}} \partial_{K_\infty} O e^I \right) \Big|_{conn,0}.$$

We used Lemma 3.13 in the second line and the easily verified algebraic identity $\tilde{\Delta}^{BV}(O e^I) = (\{I, O\} + \tilde{\Delta}^{BV} O + \frac{1}{2}\{I, I\} + \tilde{\Delta}^{BV} I) e^I$ in the last line. We have $-QO + \{I, O\} = -\{S_{kin}, O\} = 0$ since O is gauge-invariant. We have $\tilde{\Delta}^{BV} O = 0$ since O has no antifield components. Next,

$-QI + \frac{1}{2}\{I, I\} = 0$ by the master equation (3.36). We have $\tilde{\Delta}^{BV} I = 0$ since K_ϵ is symmetric in Lie-algebra indices, $\epsilon > 0$, while I is skew-symmetric in them. The final term vanishes by the following. The operation ∂_{K_∞} contracts $d\sigma$ into an $X^\dagger$ entry and 1 into an X entry (and there is also the $\mathrm{id}_\mathfrak{g}$ component one must contract). We have $\partial_{K_\infty} e^I = (\frac{1}{2}\{I, I\}_\infty + \partial_{K_\infty} I) e^I$, where $\{\cdot, \cdot\}_\infty$ involves the contraction ∂_{K_∞} (instead of ∂_{K_0}) so as to connect the input functionals. The term $\partial_{K_\infty} I$ vanishes since K_∞ is symmetric in Lie algebra indices. We thus have to consider

$$\frac{1}{2}\{I, I\}_\infty = \{I_{gh}, I_{CE}\}_\infty + \frac{1}{2}\{I_{CE}, I_{CE}\}_\infty \qquad (3.46)$$

The second term of (3.46) has an uncontracted $X^\dagger$ argument, for which $\tilde{P}_g$ will not be able to contract. Thus the operation $|_0$, which makes external leg variables zero, will annihilate all diagrams with external $X^\dagger$ legs. So we need only consider diagrams arising from the first term of (3.46). However, all such diagrams vanish using the condition $\hat{P}_g^2 = 0$. (This condition follows from $(Q_g^\dagger)^2 = 0$ and the fact that $Q_g^\dagger$ commutes with $[Q, Q_g^\dagger]$.) Indeed, one has an external A and $A^\dagger$ leg in $\{I_{gh}, I_{CE}\}_\infty$, and the placement of propagators P_g on these legs implements the operation $\hat{P}_g^2$. Since $\hat{P}_g^2 = 0$ for all compatible g, differentiating this with respect to the metric is still zero. Thus, when we consider Feynman diagrams arising from the total propagator $\tilde{P}_g$ as above, we still get that the contribution from $\{I_{gh}, I_{CE}\}_\infty$ vanishes.

Altogether, we have shown that all terms of $d_{met} \langle O \rangle_C$ vanish. This establishes gauge-invariance. $\square$

As a result of metric-independence, we can deduce that, like the exact expectation, the perturbative expectation of Wilson loops are invariant under area preserving diffeomorphisms:

Corollary 3.17. *Let $\Sigma = S^2$. Then for any f and smooth, regular γ, and choice of gauge-fixing metric, we have*

$$\langle W_{f,\gamma} \rangle_C = \langle W_{f, \Phi \circ \gamma} \rangle_C$$

if $\Phi : S^2 \to S^2$ is a diffeomorphism such that Φ preserves the area of the connected components of $S^2 \setminus \gamma$.

PROOF. Let $C_{g, d\sigma}$ denote Coulomb gauge with respect to a metric g and area form $d\sigma$ (with g compatible with $d\sigma$). Then $\langle W_{f, \Phi \circ \gamma} \rangle_{C_{g, d\sigma}} = \langle W_{f, \gamma} \rangle_{C_{\Phi^*(g), \Phi^*(d\sigma)}}$ since we can just pull back all Feynman diagrams in $\langle W_{f, \Phi \circ \gamma} \rangle_{C_g}$ by the diffeomorphism Φ. Our corollary is proven by finding a diffeomorphism Ψ_ϵ such that Ψ_ϵ fixes γ and such that $d\sigma_\epsilon := \Psi_\epsilon^*(\Phi^* d\sigma) \to d\sigma$ uniformly (in C^k with $k \geq 2$) as $\epsilon \to 0$. This is because by choosing a gauge-fixing metric g_ϵ compatible with $d\sigma_\epsilon$ such that $g_\epsilon \to g$ uniformly, we have on one hand

$$\langle W_{f, \Phi \circ \gamma} \rangle_{C_{g, d\sigma}} = \langle W_{f, \gamma} \rangle_{C_{\Psi_\epsilon^*(\Phi^*(g)), d\sigma_\epsilon}} = \langle W_{f, \gamma} \rangle_{C_{g_\epsilon, d\sigma_\epsilon}},$$

where we used independence of the gauge-fixing metric when the area form is fixed, and on the other hand

$$\langle W_{f, \gamma} \rangle_{C_{g_\epsilon, d\sigma_\epsilon}} - \langle W_{f, \gamma} \rangle_{C_{g, d\sigma}} \to 0 \qquad (3.47)$$

as a formal series in λ. Indeed, the propagator evaluated at distinct points depends continuously on the gauge-fixing metric, so that the renormalized integrals occurring in (3.47) depend continuously on the gauge-fixing metric and area form.

To construct Ψ_ϵ we proceed as follows. Thicken the image of γ to a "ribbon" R_ϵ of width ϵ (with respect to some arbitrary fixed metric), whose boundary consists of smooth disjoint simple closed curves of distance ϵ from γ. Let N_δ be a δ-thickening of ∂R_ϵ, with $\delta < \epsilon$, so that it is a union of annuli, one for each component of ∂R_ϵ.

We can define a diffeomorphism $\Psi : S^2 \to S^2$ such that (i) it maps R_ϵ and the components of $S^2 \setminus R_\epsilon$ to themselves; (ii) is the identity outside of N_δ; (iii) $\Psi^*(d\sigma') = d\sigma$ on $N_{c\delta}$ for $c < 1$ small (simply choose $\det \Psi$ appropriately on $N_{c\delta}$). Consequently, each component D of $S^2 \setminus \partial R_\epsilon$ is such that its area with respect to $d\sigma$ and $\Psi^*(d\sigma')$ differ by $O(\epsilon)$ and $d\sigma = \Psi^*(d\sigma')$ near the boundary. By the work of Moser [30], on D we can homotope the diffeomorphism $\Psi : D \to D$ (with the homotopy being identically one near ∂D) to Ψ_1 such that $\Psi_1^*(d\sigma') = d\sigma' + d\sigma(\epsilon)$, where $d\sigma(\epsilon)$ is an area form supported away from the boundary, is of order $O(\epsilon)$ in C^k, and is such that the total area of D with respect to $\Psi_1^*(d\sigma')$ and $d\sigma' + d\sigma(\epsilon)$ agree. Doing this on each component of $S^2 \setminus \partial R_\epsilon$, we have constructed the desired Ψ_ϵ. $\square$

3.4. Coulomb gauge = Holomorphic gauge.

In this section, we introduce the holomorphic gauge and use the Batalin-Vilkovisky formalism of the previous section to prove its equivalence with Coulomb gauge. The interpretation of holomorphic gauge as a gauge-fixing condition is subtle, see [32]. Nevertheless, holomorphic gauge is defined as follows.

Pick any conformal structure on S^2. (This, together with the area form is equivalent to a choice of compatible metric.) Using the resulting complex structure it induces, we can complexify the space of connections $\mathcal{A} = \Omega^1(\Sigma; \mathfrak{g})$ to $\mathcal{A}_c = \Omega^1(\Sigma; \mathfrak{g}_c)$ where $\mathfrak{g}_c$ is the complexification of $\mathfrak{g}$. The Yang-Mills action extends complex-linearly to connections belonging to $\mathcal{A}_c$ (by extending the inner product on $\mathfrak{g}$ complex-linearly). We say A is in *holomorphic gauge* if it is a differential form of type $(1,0)$, i.e. it belongs to $\Omega^{1,0}(\Sigma; \mathfrak{g}_c)$. The Yang-Mills action in holomorphic gauge becomes

$$YM(A) = \int \langle A \wedge \bar{\partial} * \bar{\partial} A \rangle, \qquad A \in \Omega^{1,0}(\Sigma; \mathfrak{g}_c). \tag{3.48}$$

Indeed, the quadratic terms in the curvature F_A vanish in holomorphic gauge.

On S^2, since there are no nontrivial holomorphic 1-forms, the pairing (3.48) is nondegenerate. Hence, the operator $\bar{\partial} * \bar{\partial} : \Omega^{1,0}(\Sigma) \to \Omega^{0,1}(\Sigma)$ is invertible and it has an integral kernel, with respect to the wedge pairing, belonging to $\Omega^{1,0}(\Sigma) \boxtimes \Omega^{1,0}(\Sigma)$. This yields yields a corresponding *holomorphic gauge propagator*

$$P_{hol} \in \Omega^{1,0}(\Sigma; \mathfrak{g}) \boxtimes \Omega^{1,0}(\Sigma; \mathfrak{g})$$

by tensoring with the identity tensor in $\mathfrak{g} \otimes \mathfrak{g}$. Explicitly, if z and w are local holomorphic coordinates on $\mathcal{C} = S^2 \setminus \{0\}$ with respect to the standard conformal structure on S^2, we have

$$P_{hol}(z, w) = \left(\frac{1}{4\pi} dz \frac{\bar{z} - \bar{w}}{z - w} dw \right) e_a e_a. \tag{3.49}$$

From the holomorphic gauge propagator, we obtain[15]

$$\langle W_{f,\gamma}\rangle_{hol} := e^{\lambda_0 \partial_{P_{hol}}} W_{f,\gamma}\Big|_0 \tag{3.50}$$

We have the following result:

Theorem 3.18. *Pick any compatible metric on S^2 determining a corresponding Coulomb gauge and holomorphic gauge. Then*

$$\langle W_{f,\gamma}\rangle_C = \langle W_{f,\gamma}\rangle_{hol}. \tag{3.51}$$

PROOF. In the same way that holomorphic gauge needs to be interpreted in terms of a real integration cycle [32], the proof of Theorem 3.18 exploits this idea in the Batalin-Vilkovisky context. Namely, we complexify the BV complex (3.22) and connect the Coulomb gauge and holomorphic gauge through a one-paramemter family of gauge-fixing operators, one which interpolates between the subspace im $* d \subset \Omega^1$ and a totally real subspace of $\Omega^{1,0}$.

We have the decompositions

$$\Omega^1 = \operatorname{im} d \oplus \operatorname{im} * d$$
$$\Omega_c^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$

with $\Omega^{1,0}$ and $\Omega^{0,1}$ the $\mp i$ eigenspaces of $*$, respectively. Concretely, we have

$$\Omega^{1,0} = \{(df + *dg) + i(-dg + *df)\}$$
$$\Omega^{0,1} = \{(df + *dg) + i(dg - *df)\}$$

given by the graphs of $\pm i* : \Omega^1 \to i\Omega^1$, where f and g denote real-valued functions. Thus, we can define a totally real subspace of $\Omega^{1,0}$ by restricting the graph of $i*$ to im $* d$:

$$\Omega_r^{1,0} := \{\alpha \in \Omega^{1,0} : \alpha = *df - idf\}.$$

We have

$$\Omega^{1,0} = \Omega_r^{1,0} \oplus i\Omega_r^{1,0}.$$

Next, we have the following complexification of the BV-complex:

$$\Omega_c^0 \xrightarrow{d} \Omega_c^1 \xrightarrow{d*d} \Omega_c^1 \xrightarrow{d} \Omega_c^2 \tag{3.52}$$

The differential Q and the Coulomb gauge-fixing operator $Q^\dagger$, defined as in (3.23), extend complex-linearly. Denote the complex (3.52) by $\mathcal{E}_c$; it consists of the terms $\mathcal{E}_c^k$ supported in degree k, $-1 \le k \le 2$.

Define the one-parameter family of chain isomorphisms $U_t : \mathcal{E}_c \to \mathcal{E}_c$, $0 \le t \le 1$, by

$$U_t|_{\mathcal{E}_c^k} = \begin{cases} \Pi_{\operatorname{im} d} + (1 + ti*)\Pi_{\operatorname{im} *d} & k = 0, 1 \\ \operatorname{id} & k \ne 1 \end{cases}$$

where the projections $\Pi_{\operatorname{im} d}$ and $\Pi_{\operatorname{im} *d}$ are complementary orthogonal projections on Ω^1 (with respect to the chosen compatible metric) extended complex linearly. Indeed, one can

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[15]Note that because there are no interactions, all Feynman diagrams are automatically connected.

check that U_t commutes with the differential Q. We have U_0 is the identity and $U := U_1$ maps im $*d \subset \mathcal{E}_c^k$ to $\Omega_r^{1,0}$, $k = 1, 2$.

Define

$$Q_t^\dagger = U_t Q^\dagger U_t^{-1}.$$

Then

$$[Q, Q_t^\dagger] = U_t[Q, Q^\dagger]U_t^{-1} = [Q, Q^\dagger]$$

since $\Delta = [Q, Q^\dagger]$ commutes with U_t. We have the t-dependent propagator

$$\hat{P}_t = Q_t^\dagger[Q, Q^\dagger]^{-1} = U_t Q^\dagger[Q, Q^\dagger]^{-1}U_t^{-1}.$$

It satisfies

$$[Q, \hat{P}_t] = \mathrm{id} - \Pi \tag{3.53}$$

just as we had in (3.24) with the Coulomb propagator.

Hence, the exact same proof as the proof for Theorem 3.4, with the ϵ-regulator inserted and then taken to zero, shows that Wilson loop expectations with respect to the propagators P_t (which are the integral kernels of $\hat{P}_t$) are independent of t. Since P_0 is the Coulomb gauge propagator, it remains to check that P_1 is the holomorphic gauge propagator. Indeed, if we do this, then

$$\left. e^{\lambda_0 \partial_{P_1}} O e^I \right|_{conn,0} = \left. e^{\lambda_0 \partial_{P_{hol}}} O \right|_0.$$

for a gauge-invariant observable O, since I vanishes whenever all the entries belong to $\Omega^{1,0}$. Moreover, for P_1, no regulator is in fact needed since the holomorphic gauge propagator (3.49) is bounded as we approach the diagonal, which renders $\langle W_{f,\gamma} \rangle_{hol}$ finite.

So we need to check that $\hat{P}_1$ inverts $\bar{\partial} * \bar{\partial} : \Omega^{1,0} \to \Omega^{0,1}$. So let $\alpha = \partial f \in \Omega^{1,0}$ for some arbitrary function f. We have

$$UQ^\dagger U^{-1}[Q, Q^\dagger]^{-1}\bar{\partial} * \bar{\partial}\alpha = UQ^\dagger U^{-1}[Q, Q^\dagger]^{-1}\bar{\partial} * \bar{\partial}\partial f$$

$$= UQ^\dagger U^{-1}[Q, Q^\dagger]^{-1}\bar{\partial}\left(-\frac{i}{2}\Delta f\right)$$

$$= -\frac{i}{2}UQ^\dagger U^{-1}\bar{\partial}f$$

$$= -\frac{i}{2}UQ^\dagger U^{-1}\left(\frac{1}{2}(1 - i*)df\right)$$

$$= -\frac{i}{2}UQ^\dagger\left(df - \frac{1}{2}i * df\right)$$

$$= \frac{i}{2}U(*df)$$

$$= \frac{i}{2}(1 + i*)(*df)$$

$$= i * \partial f$$

$$= \alpha. \quad \square$$

4. Exact Asymptotics vs. Perturbation Theory

The relation (1.5) for $\lambda\phi^4$ theories in dimensions two and three have been established, where it is known that n-point correlation functions have asymptotic expansions in small λ equal to the formal series one obtains from perturbation theory (in fact, the latter can be Borel resummed to recover the n-point functions exactly) [22, Ch 23.2]. Such results do not directly address our line of inquiry however, since for scalar field theories, both perturbative and nonperturbative calculations involve the same scalar field and therefore use identical formulations. What makes the investigation of this paper notable is that it compares two different formulations: lattice (group-valued fields) versus continuum (Lie-algebra valued) fields.

For Yang-Mills theory, the central difficulty is that while $\lim_{\lambda\to 0}\langle W_{f,\gamma}\rangle$ in two-dimensions can be evaluated by determining the asymptotics of heat kernels on Lie groups, the computation of $\langle W_{f,\gamma}\rangle_{pert}$, to all orders in λ is comparatively much harder to perform. Indeed, for Coulomb gauge, as the order in λ increases, one has an increasingly complicated set of Feynman integrals to calculate. While holomorphic gauge is simpler since the theory becomes free (i.e. the interactions I do not contribute to Feynman diagrams), the integrals one has to contribute are still highly nontrivial. This arises from G being nonabelian (the nontrivial case), since then the combinatorics of Wick contractions arising from Lie-algebraic insertions into Wilson loop operators complicates the analysis [32].

Nevertheless, we are able to obtain the following results. First, let us make more precise Conjecture 1 to include the decompactification limit. We use subscripts S^2 and $\mathbb{R}^2$ to denote the space on which exact or perturbative expectations are computed[16].

Conjecture 1'. *We have*

$$\lim_{\lambda\to 0}\langle W_{f,\gamma}\rangle_{S^2} \sim \langle W_{f,\gamma}\rangle_{S^2,hol} \tag{4.1}$$

and

$$\lim_{S^2\to\mathbb{R}^2}\lim_{\lambda_0\to 0}\langle W_{f,\gamma}\rangle_{S^2} \sim \langle W_{f,\gamma}\rangle_{\mathbb{R}^2,hol}. \tag{4.2}$$

Here $\lim_{S^2\to\mathbb{R}^2}$ denotes the decompactification limit in which the area form on the sphere increases and limits to the standard area form on $\mathbb{R}^2$. In doing so, we regard γ as a fixed subset of $S^2\setminus\{\infty\} = \mathbb{R}^2$. In (4.2), the right-hand side is by definition a formal power series in λ_0 while the left-hand side is the limit of the $\lambda_0\to 0$ asymptotic series of $\langle W_{f,\gamma}\rangle_{S^2}$ as $S^2\to\mathbb{R}^2$.

Thus, (4.1) and (4.2) correspond to cases (3) and (2), respectively, from the introduction. We now prove Theorem 2 which says that up to second order, the decompactification of the asymptotics on S^2 can be captured by holomorphic gauge on $\mathbb{R}^2$, thereby providing partial confirmation of (4.2).

Theorem 4.1. *Let γ be a simple closed curve. Then*

$$\lim_{S^2\to\mathbb{R}^2}\lim_{\lambda_0\to 0}\langle W_{f,\gamma}\rangle_{S^2} \sim \langle W_{f,\gamma}\rangle_{\mathbb{R}^2,hol} + O(\lambda_0^3). \tag{4.3}$$

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[16]On $\mathbb{R}^2$ (equipped with the standard area form), our dimensionless coupling constant λ equals λ_0. We can regard $\lambda = \lambda_0|[0,1]\times[0,1]|$, where $[0,1]\times[0,1]$ is the unit square.

PROOF. In [32], we computed the right-hand side of (4.3) for γ a simple closed curve. The result was given by a simple Gaussian matrix integral. On the other hand, Theorem 4.3 shows that the left-hand side of (4.3), to all orders in λ_0, is also given by the same Gaussian matrix integral. $\square$

The next result indicates a subtlely in the various limits and gauges that arise in two-dimensional Yang-Mills theory:

Theorem 4.2. *We have*

$$\lim_{S^2 \to \mathbb{R}^2} \lim_{\lambda_0 \to 0} \langle W_{f,\gamma} \rangle_{S^2} \not\sim \lim_{\lambda_0 \to 0} \langle W_{f,\gamma} \rangle_{\mathbb{R}^2}, \tag{4.4}$$

i.e. decompactification and small coupling asymptotics do not commute. Consequently, stochastic axial-gauge and holomorphic gauge are inequivalent on $\mathbb{R}^2$.

PROOF. Let γ be a simple closed curve. The left-hand side (4.4) is computed to all orders in (4.8). The right-hand side is determined from

$$\begin{aligned}
\langle W_{f,\gamma} \rangle_{\mathbb{R}^2} &= \int_G K_{\lambda_0 |R|}(g) f(g) dg \\
&= \left(e^{-\lambda_0 |R| \Delta/2} f \right)(1) \\
&= e^{-\lambda_0 |R| c_2(\rho)/2} \dim \rho
\end{aligned} \tag{4.5}$$

where $f = \operatorname{tr} \rho$ and $c_2(\rho)$ is the quadratic Casimir for the irreducible representation ρ. The functions (4.5) and (4.8) are not equal. In fact, they define entire functions in λ_0, and so their series expansions about $\lambda_0 = 0$ are not equal. Example 1 at the end of this section shows that (4.5) and (4.8) can disagree at second order in λ_0. Hence by Theorem 4.1, on $\mathbb{R}^2$, stochastic axial-gauge (which computes the exact expectation $\langle W_{f,\gamma} \rangle_{\mathbb{R}^2}$ via [33]) and holomorphic gauge are inequivalent. $\square$

We conclude with some explicit computations of the full small coupling asymptotics of $\langle W_{f,\gamma} \rangle_{S^2}$ for γ a simple closed curve. The final answer is remarkably simple.

Theorem 4.3. *Let γ be a simple closed curve on S^2. Define*

$$\rho = \lambda \frac{|R_1||R_2|}{|S^2|^2} \tag{4.6}$$

where R_1 and R_2 are the two regions in the complement of γ. Then for any compact Lie group G, we have as $\lambda \to 0$

$$\lim_{\lambda \to 0} \langle W_{f,\gamma} \rangle_{S^2} \sim \frac{1}{(2\pi\rho)^{d/2}} \int_{\mathfrak{g}} f(\exp(X)) e^{-|X|^2/2\rho} dX. \tag{4.7}$$

In the decompactification limit (in which $|R_2|/|S^2| \to 1$, where R_2 is the unbounded region of γ viewed as a subset of $\mathbb{R}^2$), we obtain

$$\lim_{S^2 \to \mathbb{R}^2} \lim_{\lambda_0 \to 0} \langle W_{f,\gamma} \rangle_{S^2} \sim \frac{1}{(2\pi\rho)^{d/2}} \int_{\mathfrak{g}} f(\exp(X)) e^{-|X|^2/2|R_1|\lambda_0} dX. \tag{4.8}$$

PROOF. Let H be a maximal torus of G with Lie algebra $\mathfrak{h}$. For the time being, we normalize our inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ so that the volume form it induces on $\mathfrak{g}$ is the normalized Haar measure on G. This is so that the normalized Haar measure dg occurring in the definition (2.1) of the heat kernel K_t coincides with the Riemannian volume induced from the bi-invariant metric determined by $\langle \cdot, \cdot \rangle$. (Otherwise, we will pick up awkward scale factors in what follows.) Let dY and dX denote the volume forms on $\mathfrak{g}$ and $\mathfrak{h}$ induced from the Haar measure on G and H, respectively.

Let h be a regular element of H. We have the exact formula [11, Section 7.2]

$$K_t(h) = \sum_{Y \in \exp^{-1}(h)} \frac{1}{(2\pi t)^{d/2}} \left(\det{}_Y(\exp_*) \right)^{-1/2} e^{-\frac{|Y|^2}{2t} + \frac{st}{12}}, \tag{4.9}$$

which expresses the heat kernel on a compact Lie group as a sum over geodesics. Here, s is the scalar curvature of G, $\det_Y(\exp_*)$ is the determinant of the differential of the exponential map $\exp : \mathfrak{g} \to G$ at $Y \in \mathfrak{h}$, and $|\cdot|$ is the norm with respect to the inner product on $\mathfrak{g}$. This determinant can be written as

$$\det{}_Y(\exp_*) = \frac{j(\exp(Y))}{J(Y)}, \tag{4.10}$$

where j and J are ad-invariant functions on G and $\mathfrak{g}$ given by

$$j(\exp(Y)) = \prod_{\alpha \in R_+} |e^{\alpha(Y)/2} - e^{-\alpha(Y)/2}|^2$$

$$J(Y) = \prod_{\alpha \in R_+} |\alpha(Y)|^2, \qquad Y \in \mathfrak{h}.$$

Here, R_+ is the set of positive roots of G. Thus the leading $t \to 0$ asymptotics of $K_t(h)$ for h near 1 is given by the term of (4.9) with Y of smallest length, in which case $|Y| = \mathrm{dist}(h, 1)$.

Let $t_i = \frac{\lambda |R_i|}{|S^2|}$, so that $t_1 + t_2 = \lambda$. Let W denote the Weyl group of G. Then by the Weyl integration formula

$$\frac{1}{K_{t_1+t_2}(1)} \int_G f(g) K_{t_1}(g) K_{t_2}(g) dg = \frac{1}{|W| K_{t_1+t_2}(1)} \int_H f(h) K_{t_1}(h) K_{t_2}(h) j(h) dh \tag{4.11}$$

where dh is Haar measure on H. The asymptotics of (4.11) is determined entirely by the integral in a neighborhood of 1. Using (4.9) and keeping only the leading shortest geodesic term, (4.11) as $\lambda \to 0$ is asymptotically equal to

$$\frac{(t_1+t_2)^{d/2}}{|W|(2\pi)^{d/2} t_1^{d/2} t_2^{d/2}} \int_{\mathfrak{h}} f(\exp(Y)) e^{-\frac{|Y|^2}{2t_1}} e^{-\frac{|Y|^2}{2t_2}} \left(\frac{j(\exp(Y))}{J(Y)} \right)^{-1} j(\exp(Y)) dY$$

$$= \frac{1}{|W|(2\pi\rho)^{d/2}} \int_{\mathfrak{h}} f(\exp(Y)) e^{-|Y|^2/2\rho} J(Y) dY. \tag{4.12}$$

On the other hand, this last integral is just

$$\frac{1}{(2\pi\rho)^{d/2}} \int_{\mathfrak{g}} f(\exp(X)) e^{-|X|^2/2\rho} dX \tag{4.13}$$

by the Lie algebra version of the Weyl integration formula.

Suppose now our inner product on $\mathfrak{g}$ is not the Haar inner product but c^2 times it. Observe that this scaling can be absorbed into the $*$ appearing in the Yang-Mills action, which has the effect of scaling areas by c^{-2}. In particular, we replace $\rho \mapsto \rho/c^2$ in (4.13). This is equivalent to replacing the Haar inner product with c^2 times it, i.e., our chosen inner product. Thus, (4.13) holds for the general case. $\square$

Remark 4.4. In the proof of (4.7), we neglected exponentially small contributions to (4.9) arising from non-minimal length geodesics joining the identity element of H to h. Such geodesics have nontrivial winding and we can regard their contribution to $K_t(h)$ as "instanton contributions".

In [23], additional computations are done that provide an explicit check of Conjecture 1' to all orders in λ, namely the case of products of Wilson loops formed out of concentrically nested circles. The author has also done nontrivial partial checks in the case of a figure eight loop to second order in λ.

Example 1. Let $G = SU(2)$ equipped with the bi-invariant metric induced from trace in the standard representation on $\mathfrak{g}$. We want to compute the exact asymptotics of $\langle W_{\chi_m, \gamma} \rangle$ on S^2 where χ_m is trace in the m-dimensional representation. Define the function

$$F(\rho) = e^{-\rho/4}(2 - \rho). \tag{4.14}$$

As $\lambda \to 0$, we have

$$\langle W_{\chi_m, \gamma} \rangle \sim \begin{cases} F\big((m-1)^2\rho\big) + F\big((m-3)^2\rho\big) + \cdots + F(\rho) & m \text{ even} \\ F\big((m-1)^2\rho\big) + F\big((m-3)^2\rho\big) + \cdots + F(2^2\rho) + 1 & m \text{ odd.} \end{cases} \tag{4.15}$$

Indeed, we use (4.13) or (4.12) with

$$f = \chi_m(\exp(\theta I)) = e^{i(m-1)\theta} + e^{i(m-3)\theta} + \ldots + e^{-i(m-1)\theta}$$

where we have the generators

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

for $\mathfrak{su}(2)$. So (4.12) becomes

$$\int_{\mathfrak{su}(2)} \chi_m(\exp(X)) e^{-|X|^2/2\rho} dX = \int_{\mathbb{R}^3} \chi_m(\exp(x_1 I + x_2 J + x_3 K)) e^{-|x|^2/\rho} \frac{d^3 x}{(\pi\rho)^{3/2}}$$

$$= \frac{4}{(\pi\rho^3)^{1/2}} \int_0^\infty \chi_m(\exp(\theta I)) e^{-|\theta|^2/\rho} \theta^2 d\theta.$$

This yields (4.15).

On the other hand, on $\mathbb{R}^2$ we have $\langle W_{f, \gamma} \rangle$ is given by

$$\int_G K_{\lambda_0|R|}(g) \chi_m(g) dg = e^{-\frac{(m^2-1)\lambda_0|R|}{4}} m \tag{4.16}$$

since $\Delta \chi_m = \frac{m^2-1}{2} \chi_m$. In particular, the $S^2 \to \mathbb{R}^2$ limit of (4.15), in which ρ becomes $\lambda_0|R|$, does not equal (4.16), beginning at second order in λ_0.

5. Discussion and Further Directions

We conclude with some natural questions and directions for future research.

1. Why do we obtain asymptotic series that are entire functions? The explicit asymptotic series we computed in Section 4 all defined entire power series (for f a polynomial function such as trace) in the coupling constant. Yang-Mills theory being free in axial and holomorphic gauges may make this consequence seem natural. On the other hand, the philosophy of resurgence theory [18] predicts that our series expansions should be non Borel-summable owing to the presence of instantons. At present, we have no way of reconciling this prediction with what is actually the case.

2. Compute asymptotics of more complicated Wilson loops. The asymptotic formula (4.7) and its generalization to products of Wilson loops obtained from nested, non-intersecting simple closed curves [24] suggests that perhaps similar asymptotic formulae can be obtained for Wilson loops involving more complicated curves. What is remarkable about these formulas is that they are given by simple Gaussian integrals over the Lie algebra that are of the type appearing in random matrix theory. It would be of interest to investigate how complicated one may make a Wilson loop and still continue to obtain such formulas for exact asymptotics (to all orders).

3. Consider more general topologies. In extending our results on the independence of the choice of gauge-fixing to surfaces of higher genus, the Batalin-Vilkovisky formalism would have to be carried out in the case when one has zero modes for the bosonic kinetic operator (corresponding to having a continuous moduli of flat connections). Since this difficulty is a finite-dimensional complication orthogonal to the infinite-dimensional nature of quantum field theory, we do not anticipate any fundamental obstacles. Perhaps a very interesting scenario to consider would be to consider compact surfaces with boundary. Here, boundary conditions come into play and we have not considered how the two formulations, exact and perturbative, line up in this regard.

On the other hand, in higher genus, relating the asymptotics of the exact expectation to perturbation theory may prove to be difficult, since now the set of flat connections about which to do perturbation theory becomes nontrivial. For each flat connection, we obtain a corresponding propagator formed out of the Green's operators for the Laplacian twisted by the such a flat connection. It is unclear to what extent explicit computations can be done for nontrivial flat connections.

4. Find a priori analytic relationships between the lattice and continuum formulation of quantum gauge theories. This is of course not a new question but a very difficult one, since the basic variables in the two formulations are of different natures (Lie group elements versus Lie-algebra valued 1-forms). So while an infinitesimal group element being approximated by an element in the Lie algebra provides the basis for how one discretizes the continuum theory, without precise estimates, one cannot establish any clear relation between the two. A resolution of this question would presumably shed light on some of the above questions and observations we have made.

Appendix A. Graded Vector Spaces

A (real) *graded vector space* V is a vector space together with a decomposition $V = \oplus_{i \in \mathbb{Z}} V_i$ into vector spaces V_i in degree i. If $v \in V_i$, then $|v| = i$ denotes its degree. An element is even or odd according to whether its degree is even or odd, so that we have a corresponding decomposition of $V = V_{ev} \oplus V_{odd}$ by parity. An ordinary vector space yields a graded vector space concentrated in degree zero.

For ordinary vector spaces V, one has the familiar notion of $\mathrm{Sym}^n(V)$ and $\Lambda^n(V)$, the symmetric and exterior powers of V. For graded vector spaces, one defines symmetric powers in the graded sense. Namely, let $\otimes^n V$ be the graded vector space whose graded components are

$$(\otimes^n V)_i = \bigoplus_{i_1 + \cdots + i_n = i} V_{i_1} \otimes \cdots \otimes V_{i_n}.$$

We have an action of Sym_n such that a transposition of adjacent elements acts via

$$u \otimes v \mapsto (-1)^{|u| \cdot |v|} v \otimes u.$$

Then $\mathrm{Sym}^n(V)$ is the Sym_n-invariant subspace of $\otimes^n V$ with respect to the above action. We write

$$\mathrm{Sym}(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$$

to denote the total symmetric algebra on V. Thus $\mathrm{Sym}(V)$ is a symmetric algebra and an exterior algebra (in the ordinary sense) on V_{ev} and V_{odd}, respectively.

If V is a graded vector space, then its dual space V^* is the graded vector space given by

$$V_i^* = (V_{-i})^*,$$

that is, the degree i component of V^* is the dual space of V_{-i}. In this way, the evaluation pairing

$$V^* \otimes V \to \mathbb{R}$$

is a degree zero map.

Definition A.1. Given a graded vector space V, a *function* on V is an element of $\mathrm{Sym}(V^*)$. A *vector field* on V is an element of $\mathrm{Sym}(V^*) \otimes V$.

For V an ordinary vector space, the above coincides with the ordinary notion of a polynomial function or polynomial vector field.

A (linear) map $f : V \to V'$ of graded vector spaces has degree p if $f(V_i) \subset V'_{i+p}$. If the degree is not specified, it is understood to be degree zero. The commutator of two maps $f, g : V \to V$ of degrees $|f|$ and $|g|$ is defined using the appropriate sign rule:

$$[f, g] = fg - (-1)^{|f||g|} gf.$$

A.1. **Directional derivatives.** The space $(V^*)^{\otimes n}$ denote the space of n-multilinear maps from $V^{\otimes n}$ to $\mathbb{R}$. It has a natural action of Sym_n induced from the one on $V^{\otimes n}$. For $v \in V$, define the contraction operator

$$\partial_v : (V^*)^{\otimes n} \to (V^*)^{\otimes (n-1)}$$

$$v_1^* \otimes \cdots \otimes v_n^* \mapsto \sum_i (-1)^{|v|(|v_1| + \cdots + |v_{i-1}|)} v_i^*(v) \left(v_1^* \otimes \cdots \otimes v_{i-1}^* \otimes v_{i+1}^* \otimes \cdots \otimes v_n^* \right). \quad \text{(A.1)}$$

In other words, ∂_v is the directional derivative with respect to v, where in the graded setting, it is a derivation of degree $|v|$ using the above signed Leibniz rule.

More generally, given an element $K = u \otimes v \in V^{\otimes 2}$, we can define the operation

$$\partial_K = \frac{1}{2} \partial_v \partial_u. \tag{A.2}$$

This operation extends bilinearly to any $K \in V^{\otimes 2}$ and depends only on the component of K in $\mathrm{Sym}_2(V)$.

One can consider contractions using "non-constant coefficient" vector fields, i.e., elements of $\mathrm{Sym}(V^*) \otimes V$. If $fv \in \mathrm{Sym}(V^*) \otimes V$, with $f \in \mathrm{Sym}(V^*)$, then

$$\partial_{fv} = f \partial_v$$

Likewise, we have

$$\partial_{vf} = (-1)^{|f||v|} f \partial_v \tag{A.3}$$

using the usual sign rules.

A.2. **Infinite-dimensional case.** The above considerations generalize to the infinite-dimensional setting needed for quantum field theory. Instead of a finite-dimensional graded vector space, we instead have the space of sections $\mathcal{E}$ of a graded vector bundle E over a smooth manifold M. We leave the regularity of the elements of $\mathcal{E}$ unspecified (i.e. whether they are smooth or distributional), it usually being clear from the context. The dual space $\mathcal{E}^*$ of the smooth elements of $\mathcal{E}$ consists of distributions on M valued in the dual bundle of E. Multilinear functionals on $\mathcal{E}$ are elements of $\mathcal{E}^* \otimes \cdots \otimes \mathcal{E}^*$, where the tensor product is completed[17]. Given $v \in \mathcal{E}$, one defines $\partial_v : (\mathcal{E}^*)^{\otimes n} \to (\mathcal{E}^*)^{\otimes n-1}$ as above. Likewise for $K \in \mathcal{E}^{\otimes 2}$, which we regard as an integral kernel of an operator (defined with respect to a pairing on $\mathcal{E}$, see Section 3.3), we can define ∂_K as above. If K is not smooth, ∂_K may be ill-defined when evaluated on a multilinear functional. In particular, when K arises as the integral kernel of a differential operator, one can obtain divergent Feynman integrals from the contractions ∂_K applied to local functionals. An element of multilinear map $(\mathcal{E}^*)^{\otimes n}$ is said to be *local* if is given by the integral of a polydifferential function of $\mathcal{E}$ over M.

Appendix B. Wick's Theorem

Wick's Theorem comes in two cases: bosonic and fermionic. The first case gives us a combinatorial formula for the integration of monomials against Gaussian measures. We assume the reader is familiar with this result and only record it here for notational purposes. Consider $\mathbb{R}^d$ with the standard inner product $(\cdot, \cdot)$ and let $A = A_{ij}$ be a symmetric nondegenerate $d \times d$ matrix. It determines a bilinear form $(\cdot, A \cdot)$ and a normalized Gaussian measure $d\mu_A = \left(\frac{\det A}{(2\pi)^d} \right)^{1/2} e^{-(x, Ax)/2} d^d x$.

Lemma B.1. *(Bosonic Wick's Theorem) Consider the monomial $f(x) = x^{i_1} \cdots x^{i_{2n}}$. Then*

$$\int d\mu_A f(x) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} A^{i_{\sigma(1)} i_{\sigma(2)}} \cdots A^{i_{\sigma(2n-1)} i_{\sigma(2n)}}. \tag{B.1}$$

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[17] We have $C^\infty(M_1 \times M_2) = C^\infty(M_1) \otimes C^\infty(M_2)$, where the right-hand side is completed in the sense of nuclear Frechet spaces. This readily extends to the case of smooth sections of a vector bundle, so that $\Gamma(E_1 \boxtimes E_2) = \Gamma(E_1) \otimes \Gamma(E_2)$.

Here A^{ij} denotes the inverse matrix of A_{ij}.

As is well-known, the sum on the right-hand-side has an elegant description in terms of Feynman diagrams. For further details, see e.g. [15, 34].

Next, consider the fermionic case of Wick's Theorem. For this we have to introduce the notion of integration over fermionic (odd) variables. Let V be an odd vector space spanned by $\xi_1, \ldots, \xi_d$. Then we can define the partial integration operator $\int d\xi_i$ by

$$\int d\xi_i = \partial_{\xi_i},$$

where ∂_{ξ_i} is defined as in (A.1). Thus,

$$\int d\xi_i \xi_i = 1$$

and more generally

$$\int d\xi_i \xi_i f(\xi) = f(\xi)$$

$$\int d\xi_i f(\xi) = 0.$$

if $f(\xi)$ does not depend on ξ_i. We write $\int d\xi_d \ldots d\xi_1$ as shorthand for $\int d\xi_d \cdots \int d\xi_1$. Thus,

$$\int d\xi_d \ldots d\xi_1 \xi_1 \ldots \xi_d = 1.$$

Suppose we have an even number of Grassman variables $d = 2m$. Given a nondegenerate skew-symmetric matrix A_{ij}, we get the bilinear expression $\xi_i A_{ij} \xi_j / 2$, which we abbreviate as $(\xi, A\xi)/2$. Letting $d\mu = d\xi_1 \ldots d\xi_d$, we have the following:

Lemma B.2. *We have*

$$\int d\mu e^{-(\xi, A\xi)/2} = \mathrm{Pfaf}(A).$$

Let

$$d\mu_A = \mathrm{Pfaf}(A)^{-1} d\mu e^{-(\xi, A\xi)/2}$$

denote the normalized "Gaussian measure" on odd variables. We can integrate (polynomial) functions in the ξ_i against this density. We have the identity

$$A^{ik} \partial_{\xi_k} (e^{-(\xi, A\xi)/2} f(\xi)) = e^{(-\xi, A\xi)/2} \big(-\xi_i f(\xi) + A^{ik} \partial_{\xi_k} f(\xi) \big)$$

Letting $f(\xi) = \xi_j$, we conclude that

$$\int d\mu_A \xi_i \xi_j = A^{ij}. \tag{B.2}$$

Iteration of this yields the following formula:

Lemma B.3. *(Fermionic Wick's Theorem) Consider the monomial* $f(\xi) = \xi_{i_1} \cdots \xi_{i_{2n}}$. *Then*

$$\int d\mu_A f(\xi) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} (-1)^\sigma A^{i_{\sigma(1)} i_{\sigma(2)}} \cdots A^{i_{\sigma(2n-1)} i_{\sigma(2n)}}. \tag{B.3}$$

It is often the case that the set of odd variables is partitioned into two separate sets, $\omega_1 \ldots, \omega_m$ and their corresponding "conjugate" variables $\omega_1^*, \ldots, \omega_m^*$. Moreover, we are given the bilinear expression $\omega_i^* B_{ij} \omega_j$ with B_{ij} is an arbitrary matrix, which we abbreviate by $\omega^* B \omega$. One can think of this as letting $\xi_k = \omega_k^*$ and $\xi_{m+k} = \omega_k$ for $1 \le k \le m$ and

$$A = \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix} \tag{B.4}$$

in the above. Let $d\mu = d\omega_1^* d\omega_1 \cdots d\omega_m^* d\omega_m$.

Lemma B.4.

$$\int d\mu e^{-\omega^* B \omega} = \det B.$$

Since

$$\mathrm{Pfaf}(A) = (-1)^{\binom{2m}{2}} \det(B)$$

$$d\xi_1 \cdots d\xi_{2m} = (-1)^{\binom{2m}{2}} d\omega_1^* d\omega_1 \cdots d\omega_m^* d\omega_m,$$

the formulas in Lemmas B.2 and B.4 agree.

We suppose B_{ij} is invertible. Let $d\mu_B = \frac{1}{\det B} d\mu e^{-(\xi^*, B\xi)}$. From

$$B^{ik} \partial_{\omega_k^*}(e^{-(\omega^*, B\omega)} P) = e^{(-\omega^*, B\omega)}\Big(-\omega_i P + B^{ik} \partial_{\omega_k^*} P \Big),$$

we have

$$\int d\mu_B \omega_i \omega_j^* = B^{ij},$$

or in other words

$$\int d\mu_B \omega_i^* \omega_j = -B^{ij}. \tag{B.5}$$

One can attribute this minus sign to the minus sign that occurs in

$$A^{-1} = -\begin{pmatrix} 0 & -B^{-1} \\ B^{-1} & 0 \end{pmatrix} \tag{B.6}$$

for A of the form (B.4).

Remark B.5. The above considerations explain the convention that fermionic loops in quantum field theoretic computations are weighted with a minus sign. This is a basis dependent statement, however. One should regard (B.3) as fundamental, with (B.5) a consequence of a particular (albeit common) parametrization.

One can unify the bosonic and fermionic cases of Wick's Theorem using the framework of graded vector spaces. Let $A(\cdot, \cdot)$ be a symmetric pairing on a graded vector space V, i.e., an element of $\mathrm{Sym}^2(V^*)$, which is nondegenerate. Such a pairing determines a dual pairing P on V^* which is an element of $\mathrm{Sym}^2(V)$. We call P the *propagator*.

If we pick a basis v_i (of homogeneous degree elements) for V, then if $A_{ij} = A(v_i, v_j)$, we have $P = A^{ij} v_i v_j$. Define

$$d\mu_A = c_A \prod dv_i e^{-A_{ij} v_i v_j / 2}$$

with c_A chosen so that so that $\int d\mu_A = 1$. We have the following:

Lemma B.6. *(Wick's Theorem, unified version) For f a (polynomial) function on V,*

$$\int d\mu_A f(x) = (e^{\partial_P} f)(0). \tag{B.7}$$

Here, $\partial_P : \mathrm{Sym}^n(V^*) \to \mathrm{Sym}^{n-2}(V^*)$ is the *Wick contraction* operator, defined as in Section A.1, and

$$e^{\partial_P} = \sum_{n=0}^{\infty} \frac{(\partial_P)^n}{n!}$$

is the sum over all Wick contractions weighted with the appropriate symmetry factor. The right-hand side of (B.7) is evaluated at zero so that the maximal number of Wick contractions have been made (in the Feynman-diagrammatic picture, we only sum over vacuum diagrams, i.e. those without external legs).

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